Admissibilty for Volterra systems with scalar kernels

B.H. Haak

The purpose of this article is to present conditions for the admissibility of observation operators to parabolic Volterra equations, that is, we consider the 'observed' system

$$\begin{aligned} x(t) &= x_0 + \int_0^t a(t-s)\, Ax(s)\, ds, \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $t \geq 0$. Here, the operator A is supposed to be a closed operator with dense domain on a Banach space X, $x_0 \in X$, the kernel function $a \in L^1_{\text{loc}}$ is supposed to be of sub-exponential growth and 1-regular, and it is assumed that (1) is parabolic. In Prüss [7] it is shown that under these assumptions, equation (1) admit a unique *solution family*, i.e. a family of bounded linear operators $(S(t))_{t\geq 0}$ on X.

For some results we need in addition that $-A$ a sectorial operator of type $\omega \in (0, \pi)$ or that the kernel a is *sectorial of angle* $\theta \in (0, \pi)$. The kernel a is called *sectorial of angle* $\theta \in (0, \pi)$ if

$$\widehat{a}(\lambda) \in S_\theta \qquad \text{for all } \lambda \text{ with positive real part.}$$

In particular, when $-A$ and a are both sectorial in the respective sense with angles that sum up to a constant strictly inferior to π, the Volterra equation is parabolic.

The operator C is supposed to be an operator from X into another Banach space Y that acts as a bounded operator from $X_1 \to Y$ where $X_1 = \mathcal{D}(A)$ is endowed by the graph norm of A. In order to guarantee that the output function lies locally in L_2 we are interested in the following property.

Definition 1. A bounded linear operator $C : X_1 \to Y$ is called *finite-time admissible* for the Volterra equation (1) if there are constants $\eta, K > 0$ such that

$$\left(\int_0^t \|CS(r)x\|^2 \, dr \right)^{\frac{1}{2}} \leq K e^{\eta t} \|x\|$$

for all $t \geq 0$ and all $x \in \mathcal{D}(A)$.

The notion of admissible observation operators is well studied in the literature for Cauchy systems, that is, $a \equiv 1$, see for example [3], [8], and [9]. Admissible observation operators for Volterra systems are studied in [2], [4], [5] and [6].

The Laplace transform of S, denoted by H, is given by

$$H(\lambda)x = \frac{1}{\lambda}(I - \hat{a}(\lambda)A)^{-1}x, \qquad \operatorname{Re} \lambda > 0.$$

Our first main result, Theorem 2 provides a subordination argument to obtain admissibility for the observed Volterra equation from the admissibility of the observation operator for the underlying Cauchy problem.

Theorem 2. Let A generate an exponentially stable strongly continuous semigroup $(T(t))_{t\geq 0}$ and let $C : X_1 \to Y$ be bounded. Further we assume that the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is of sub-exponential growth, 1-regular and sectorial of angle $\theta < \pi/2$. Then finite-time admissibility of C for the semigroup $(T(t))_{t\geq 0}$ implies that of C for the solution family $(S(t))_{t\geq 0}$.

This allows a large number of corollaries, based on positive results for the Weiss conjecture. Here we only mention the following.

Corollary 3. Assume in addition to the hypotheses of the theorem that A admits a Riesz basis of eigenfunctions (e_n) on a Hilbert space X with corresponding eigenvalues λ_n. If $Y = \mathbb{C}$ and if

$$\mu = \sum_n |Ce_n|^2 \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$, then C is finite-time admissible for the solution family $(S(t))_{t\geq 0}$.

This corollary improves a direct Carleson measure criterion from Haak, Jacob, Partington and Pott [2]. Our second main result provides a sufficient condition for admissibility.

Theorem 4. Assume that A is a closed operator with dense domain on X, the kernel function $a \in L^1_{\text{loc}}$ is of sub-exponential growth, 1-regular, and (1) is parabolic. Let $C : X_1 \to Y$ be bounded and assume that for some $\alpha > \frac{1}{2}$,

$$\sup_{r>0} \left\| \ (1 + \log^+ r)^\alpha r^{\frac{1}{2}} C H(r) \right\| < \infty. \tag{2}$$

Then C is finite-time admissible for $(S(t))_{t \geq 0}$.

The obtained results are applied to time-fractional diffusion equations of distributed order and are compared with other results on Volterra systems known so far.

This is a joint work with B. Jacob.

References

[1] B. Haak, B. Jacob, *Observation of Volterra systems with scalar kernels*, Journal of Integral Equations and Applications, to appear.

[2] B. Haak, B. Jacob, J.R. Partington, and S. Pott, Admissibility and Controllability of diagonal Volterra equations with scalar inputs, *J. Differential Equations*, 246 (2009), 4423-4440.

[3] B. Jacob and J.R. Partington, Admissibility of control and observation operators for semigroups: a survey, in J.A. Ball, J.W. Helton, M. Klaus and L. Rodman: Current Trends in Operator Theory and its Applications, Proceedings of IWOTA 2002, Operator Theory: Advances and Applications, Vol. 149, Birkhäuser (2004), 199-221.

[4] B. Jacob and J.R. Partington, Admissible control and observation operators for Volterra integral equations, *Journal of Evolution Equations*, 4 (2004), 333-343.

[5] B. Jacob and J.R. Partington, A resolvent test for admissibility of Volterra observation operators, *J. Math. Anal. and Appl.*, 332 (2007), 346-355.

[6] M. Jung, Admissibility of control operators for solution families to Volterra integral equa- tions, *SIAM J. Control Optim.*, 38 (2000), 1323-1333.

[7] J. Prüss, *Evolutionary integral equations and applications*, vol. 87 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1993.

[8] O. Staffans, *Well-Posed Linear Systems*, no. 103 in Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2005.

[9] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.*, 65 (1989), pp. 17-43.