Hahn-Banach type theorems for normed modules

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Let A be a normed algebra, $\mathcal{K}$ some class of left normed A-modules. A left normed A-module Z is called *extremely $\mathcal{K}$-injective* if, for every A-module Y and its submodule X, every bounded morphism $X \to Z$ can be extended to a morphism $Y \to Z$ of the same norm. (Thus, Z plays the role of $\mathbb{C}$ in the classical Hahn-Banach theorem).

In the following theorem we consider, as A, the algebra $\mathcal{B}(L)$ of all bounded operators on an infinite-dimensional Hilbert space L, and, as $\mathcal{K}$, the class of left Ruan modules (those X with the property $\|u + v\| \leq \sqrt{\|u\|^2 + \|v\|^2}$, provided $u, v \in X$ satisfy $u = P \cdot u, v = Q \cdot v$ for some mutually orthogonal projections $P, Q \in \mathcal{B}(L)$). These modules were introduced in connection with attempts to obtain a transparent proof of the Arveson-Wittstock Extension Theorem, one of fundamental principles of quantum functional analysis.

Theorem 1. *Let H be an arbitrary Hilbert space, and $L \otimes H$ a Hilbert A-module with the outer multiplication $a \cdot (\xi \otimes \eta) := a(\xi) \otimes \eta$. Then such a module is extremely $\mathcal{K}$-injective.*

This theorem, combined with some general facts about Ruan modules, gives, as an easy corollary, Arveson-Wittstock Theorem.

Later Wittstock generalized and strengthened the formulated theorem in several directions. In particular, he proved that, with A and $\mathcal{K}$ as above, every dual to a Ruan module is $\mathcal{K}$-injective.

Turn to the opposite class of commutative algebras. What about modules over one of the simplest, the algebra c_0 of vanishing sequences? The following theorem describes extremely $\mathcal{K}$-injective modules within a certain reasonable class of c_0-modules. Namely, we call a c_0-module Z *homogeneous*, if, for $z', z'' \in Z$, the equalities $\|p^n \cdot z'\| = \|p^n \cdot z''\|; n = 1, 2, \ldots,$ where $p^n = (0, \ldots, 0, 1, 0, \ldots)$, imply $\|z'\| = \|z''\|$.

Theorem 2. *Let $\mathcal{K}$ be a class of homogeneous* c_0*-modules, and* Z *is a non-degenerate homogeneous* c_0*-module. Then the module* Z^* *is extremely* $\mathcal{K}$*-injective if, and only if, for every* $n = 1, 2, ..,$ *the normed space* $\{p^n \cdot z; z \in Z\}$ *is, up to an isometric isomorphism, a dense subspace of* $L_1(\Omega_n)$ *for some measure space* Ω_n.

In particular, all c_0-modules $l_p; 1 \leq p \leq \infty$ are extremely $\mathcal{K}$-injective.

The condition of the non-degeneracy of Z can not be omitted: $Z := l_\infty$ provides the relevant counter-example.

One of basic tools of the proof of both theorems is the algebraic "law of adjoint associativity", properly modified to serve in functional analysis.