Total positivity and the Riemann zeta-function

O. Katkova

The sequence $\{a_k\}_{k=0}^{\infty}$ is called m-times positive, $m \in \mathbf{N}$, (totally positive), if all minors of order $\leq m$ (of any order) of the infinite matrix $A = (a_{j-i}), i, j = 0, 1, 2, \dots$ ($a_k = 0$ for $k < 0$) are nonnegative. The class of corresponding generating functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is denoted by $PF_m(PF_\infty)$.

In 1953 Aissen, Schoenberg, Whitney and Edrei obtained the full description of functions $f \in PF_\infty$: $f(z) = Cz^n e^{\gamma z} \prod_{k=1}^{\infty} \frac{1+\alpha_k z}{1-\beta_k z}$, where $C \geq 0, n \in \mathbf{Z}, \gamma \geq 0$, $\alpha_k \geq 0, \beta_k \geq 0, \sum(\alpha_k + \beta_k) < \infty$.

We consider the following function

$$\xi_1(z) = \frac{1}{2}(z - \frac{1}{4})\pi^{-\sqrt{z}/2-1/4}\Gamma(\sqrt{z}/2 + 1/4)\zeta(\sqrt{z} + 1/2),$$

where ζ is the Riemann ζ-function.

The function ξ_1 is an entire function of order $\frac{1}{2}$ and the Riemann Hypothesis is equivalent to the statement that ξ_1 has only real negative zeros. By Theorem of Aissen, Schoenberg, Whitney and Edrei it means that $\xi_1 \in PF_\infty = \cap_{m=1}^{\infty} PF_m$.

We obtain the following result.

Theorem. For all $m \in \mathbf{N}$ the sequence of coefficients of ξ_1 is asymptotically m-times positive, that is for all $m \in \mathbf{N}$ there exists a positive integer N such that all minors of matrix $A_N = (a_{N+j-i}), i = 0, 1, 2, \dots, m-1, \quad j = o, 1, \dots$ ($a_k = 0$ for $k < 0$) are nonnegative.