Hypercomplex representations of the Heisenberg group: p-mechanics

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This is a report on the ongoing work on unification of quantum and classical mechanical formalism, known as p-mechanics. Complex valued representations of the Heisenberg groups naturally provide a natural framework for quantum mechanics. This is the most fundamental example of the Kirillov orbit method and geometrical quantisation technique.

Following the pattern we consider representations of the Heisenberg group which are induced by hypercomplex characters of its centre. Besides complex numbers (which correspond to the elliptic case, $i^2 = -1$) there are two other types of hypercomplex numbers: dual (parabolic, $i^2 = 0$) and double (hyperbolic, $i^2 = 1$).

To describe dynamics of a physical system we use a universal equation based on inner derivations of the convolution algebra. The complex valued representations produce the standard framework for quantum mechanics with the Heisenberg dynamical equation.

The representations with value in dual numbers provide a convenient description of the classic mechanics. For this we do not take any sort of semiclassical limit, rather the nilpotency of the imaginary unit ($i^2 = 0$) removes the vicious necessity to consider the Planck constant tending to zero. The dynamical equation takes the Hamiltonian form. One of mathematical models uses $\mathbb{R}$-min ($\mathbb{R}$-max) algebras, which are studied within tropical mathematics and Maslov's dequantisation.

The double number valued representations, with the imaginary unit $i^2 = 1$, is a natural source of hyperbolic quantum mechanics developed recently by A. Khrennikov. The universal dynamical equation employs hyperbolic commutator. This can be seen as a version of the Moyal bracket based on the hyperbolic sine function.

The approach provides not only with three different types of dynamics,

it also generates the respective rules for addition of probabilities. For example, the quantum interference is the consequence of the same structure which direct the Heisenberg equation. The absence of an interference (a wave behaviour) in the classical is again the consequence the nilpotency of the imaginary unit. Double numbers creates the hyperbolic law of additions of probabilities which were extensively investigates by A. Khrennikov.

The work clarifies foundations of quantum and classical mechanics. It also hinted that hyperbolic counterpart is (at least theoretically) as natural as classical and quantum mechanics are. The approach provides a framework for description of aggregate system which have say both quantum and classical components. This can be used to model quantum computers with classical terminals.