Relaxing linear matrix inequalities noncommutatively

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Given linear matrix inequalities (LMIs) L_1 and L_2 it is natural to ask:

(Q_1) when does one dominate the other, that is, does $L_1(X) \succeq 0$ imply $L_2(X) \succeq 0$?

(Q_2) when are they mutually dominant, that is, when do they have the same solution set?

In this talk we describe a natural relaxation of an LMI, based on substituting matrices for the variables x_j. With this relaxation, the domination questions (Q_1) and (Q_2) have elegant answers. Assume there is an X such that $L_1(X)$ and $L_2(X)$ are both positive definite, and suppose the positivity domain of L_1 is bounded. For our "matrix variable" relaxation a positive answer to (Q_1) is equivalent to the existence of matrices V_j such that

$$L_2(x) = V_1^* L_1(x) V_1 + \cdots + V_\mu^* L_1(x) V_\mu. \tag{A$_1$}$$

As for (Q_2), L_1 and L_2 are mutually dominant if and only if, up to certain redundancies, L_1 and L_2 are unitarily equivalent.

Algebraic certificates for positivity, such as (A_1) for linear polynomials, are typically called Positivstellensätze. We shall also explain how to derive a Putinar-type Positivstellensatz for polynomials with a cleaner and more powerful conclusion under the stronger hypothesis of positivity on an underlying bounded domain of the form $\{X \mid L(X) \succeq 0\}$.

An observation at the core of this talk is that the relaxed LMI domination problem is equivalent to a classical problem in operator algebras. Namely, the problem of determining if a linear map from a subspace of matrices to a matrix algebra is *completely positive*.

The talk is based on joint work with J.W. Helton and S. McCullough; see http://arxiv.org/abs/1003.0908