On generalized Schoenberg theorem

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This talk is devoted to the generalization of the theorem, first proved by I.J. Schoenberg, concerning the variational-diminishing property of a totally positive matrix.

Let A be a linear operator, acting in the space $\mathbb{R}^n$. In this case the operator $\wedge^j A$ $(j = 1, \dots, n)$, i.e. the j-th exterior power of the operator A, acts in the space $\wedge^j \mathbb{R}^n = \mathbb{R}^{C_n^j}$. A set $K \subset \mathbb{R}^n$ is called *a proper cone*, if it is a convex cone (i.e. for any $x, y \in K$, $\alpha \geq 0$ we have $x + y$, $\alpha x \in K$), is pointed (i.e. $K \cap (-K) = \{0\}$), closed and full (i.e. $\mathrm{int}(K) \neq \emptyset$). A linear operator A is called *generalized totally positive* if it leaves invariant a proper cone $K_1 \subset \mathbb{R}^n$, and for every j $(1 < j \leq n)$ its j-th exterior power $\wedge^j A$ leaves invariant a proper cone $K_j \subset \mathbb{R}^{C_n^j}$.

A closed subset $T \subset \mathbb{R}^n$ is called *a cone of rank* k $(0 \leq k \leq n)$, if for every $x \in T$, $\alpha \in \mathbb{R}$ the element $\alpha x \in T$ and there is at least one k-dimensional subspace and no higher dimensional subspaces in T.

Let the operator A be generalized totally positive. Then under some additional conditions we can prove the existence of n cones $T_1, \dots, T_n$, each of which is invariant for the operator A, and the rank of the jth cone T_j is equal to j. The inverse is also true: if for every $j = 1, \dots, n$ the operator A leaves invariant a cone T_j of rank j, then under some additional conditions we can prove the existence of n proper cones $K_1, \dots, K_n$, such that the jth cone K_j is invariant for the jth exterior power $\wedge^j A$ of the initial operator A.

References.

1. I.J. Schoenberg, *Über variationsvermindernde lineare Transformationen.* Math. Z. **32** (1930), 321-328.

2. A. Pinkus, *Totally positive matrices.* Cambridge University Press, 2010.