Two-dimensional Hamiltonian systems with two singular endpoints

M. Langer (Part one) and H. Woracek (Part two)

Consider a two-dimensional Hamiltonian system of the form

$$y'(x) = zJH(x)y(x), \quad x \in (0, \infty), \tag{1}$$

which is in Weyl's limit point case at the endpoint ∞. If at the left endpoint 0 limit circle case prevails, the spectral theory of this system is well understood. The associated (minimal) differential operator is symmetric with defect index $(1, 1)$, and the notion of the Titchmarsh-Weyl coefficient q_H associated with (1) allows to construct a Fourier transform onto an $L2$-space with respect to a scalar valued positive measure μ (appropriately including a possible point mass at infinity). The measure μ is thereby obtained from the Herglotz-integral representation of q_H; in particular, it satisfies $\int_{\mathbb{R}}(1 + t2)^{-1}\, d\mu < \infty$. An inverse spectral theorem, due to L.de Branges, holds true. It states that each such measure appears as the spectral measure of a system of this kind, and that the system (1) can –at least in theory– be recovered from μ.

If also at the endpoint 0 limit point case takes place, the situation changes. The (minimal) differential operator is already selfadjoint. The classical theory leads, via the Titchmarsh-Kodaira formula, to a Fourier transform onto an $L2$-space with respect to a 2×2-matrix valued measure. In general this result cannot be improved; the minimal operator may in general have spectral points with spectral multiplicity 2.

Nevertheless, there exist classes of Hamiltonians which are in the limit point case at both endpoints, but still behave well in the sense that a Fourier transform onto a space of scalar valued functions can be constructed. One such class was already given by L.de Branges, however, no inverse results were proved. Other previous work in this direction focuses mainly on the case of Sturm-Liouville equations with singular potential (which can be regarded as a particular case of Hamiltonian systems), and also deals mainly with the direct spectral problem.

We present a class of Hamiltonians, being in the limit point case at both endpoints, for which the direct and inverse spectral problems can be solved in a complete and satisfactory way. It is defined by means of growth restrictions at the left endpoint. Concerning the direct spectral problem, we show that each Hamiltonian H of this class gives rise to a Fourier transform onto a space $L2(\mu)$ where μ is a scalar valued positive measure such that $\int_{\mathbb{R}}(1+t2)^{-n}\,d\mu < \infty$ for some $n \in \mathbb{N}$. Probably more notable, we show that each measure subject to this growth restriction arises in this way, and that H can (in general again unconstructively) be recovered from μ.

Although the problem under consideration as well as the final solution to it is purely 'positive definite', our approach proceeds via Pontryagin space theory. We use an indefinite version of Hamiltonian systems, generalized Nevanlinna functions, and selfadjoint relations in Pontryagin spaces. We find it particularly appealing that, after a (quite significant) detour through the indefinite world, a classical solution to a classical problem is obtained.