The index formula and the spectral shift function for relatively trace class perturbations

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We compute the Fredholm index, $\operatorname{index}(\boldsymbol{D}_{\boldsymbol{A}})$, of the operator $\boldsymbol{D}_{\boldsymbol{A}} = (d/dt) + \boldsymbol{A}$ on $L^2(\mathbb{R};\mathcal{H})$ associated with the operator path $\{A(t)\}_{t=-\infty}^{\infty}$, where $(\boldsymbol{A}f)(t) = A(t)f(t)$ for a.e. $t \in \mathbb{R}$, and appropriate $f \in L^2(\mathbb{R};\mathcal{H})$, via the spectral shift function $\xi(\,\cdot\,;A_+,A_-)$ associated with the pair (A_+,A_-) of asymptotic operators $A_\pm = A(\pm\infty)$ on the separable complex Hilbert space $\mathcal{H}$ in the case when $A(t)$ is generally an unbounded (relatively trace class) perturbation of the unbounded self-adjoint operator A_-.

We derive a formula (an extension of a formula due to Pushnitski) relating the spectral shift function $\xi(\,\cdot\,;A_+,A_-)$ for the pair (A_+,A_-), and the corresponding spectral shift function $\xi(\,\cdot\,;\boldsymbol{H}_2,\boldsymbol{H}_1)$ for the pair of operators $(\boldsymbol{H}_2,\boldsymbol{H}_1) = (\boldsymbol{D}_{\boldsymbol{A}}\boldsymbol{D}_{\boldsymbol{A}}^*, \boldsymbol{D}_{\boldsymbol{A}}^*\boldsymbol{D}_{\boldsymbol{A}})$ in this relative trace class context,

$$\xi(\lambda;\boldsymbol{H}_2,\boldsymbol{H}_1) = \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu;A_+,A_-)\,d\nu}{(\lambda-\nu^2)^{1/2}} \text{ for a.e. } \lambda > 0.$$

This formula is then used to identify the Fredholm index of $\boldsymbol{D}_{\boldsymbol{A}}$ with $\xi(0;A_+,A_-)$. In addition, we prove that $\operatorname{index}(\boldsymbol{D}_{\boldsymbol{A}})$ coincides with the spectral flow $\operatorname{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty})$ of the family $\{A(t)\}_{t\in\mathbb{R}}$ and also relate it to the (Fredholm) perturbation determinant for the pair (A_+,A_-):

$$\operatorname{index}(\boldsymbol{D}_{\boldsymbol{A}}) = \operatorname{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) = \xi(0;A_+,A_-) = \pi^{-1}\operatorname{Im}\,\ln(\det{}_{\mathcal{H}}(A_+A_-^{-1})).$$

We also provide some applications in the context of supersymmetric quantum mechanics to zeta function and heat kernel regularized spectral asymmetries and the eta-invariant.

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