Dirichlet problems for Ornstein-Uhlenbeck operators in subsets of Hilbert spaces

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We consider a family of self-adjoint Ornstein–Uhlenbeck operators $\mathcal{L}_\alpha$ in an infinite dimensional Hilbert space H having the same gaussian invariant measure $\mu = \mathcal{N}_Q$,

$$\mathcal{L}_\alpha \varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[Q^{1-\alpha} D^2 \varphi(x)] - \frac{1}{2}\langle x, Q^{-\alpha} D\varphi(x)\rangle,$$

where $Q \in \mathcal{L}(H)$ is a symmetric positive operator with finite trace, and $0 \le \alpha \le 1$.

We study the Dirichlet problem for the equation $\lambda\varphi - \mathcal{L}_\alpha \varphi = f$ in a closed set $K \subset H$, with $f \in L^2(K, \mu)$. Its variational solution, trivially provided by the Lax—Milgram theorem, can be represented by means of the transition semigroup stopped to K, as in finite dimensions.

We address two problems: 1) the meaning of the Dirichlet boundary condition; 2) the regularity of the solution φ (which belongs to a Sobolev space $W^{1,2}_\alpha(K, \mu)$ by definition) of the Dirichlet problem.

Concerning the boundary condition we consider both irregular and regular boundaries. In the first case we content to have a solution whose null extension outside K belongs to $W^{1,2}_\alpha(H, \mu)$. In the second case we exploit the Malliavin's theory of surface integrals, to give a meaning to the trace of φ at ∂K and to show that it vanishes, as it is natural.

Concerning regularity, we can prove interior $W^{2,2}_\alpha$ regularity results. Regularity up to the boundary is much more complicated; however we have some partial results. For instance, we can treat the case $\alpha = 0$ for halfspaces $K = \{x \in H : \langle b, x\rangle = 1\}$ with $b \in H$, $\|b\| = 1$.

The talk is based on a joint work with G. Da Prato.