Self-adjoint analytic operator functions: Inner linearization and factorization of the restriction to a spectral subspace

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Let $\Delta = [a, b]$ be a real interval, D be a neighborhood of Δ and $\mathcal{H}$ be a Hilbert space. We consider a $\mathcal{L}(\mathcal{H})$-valued function $A(z)$ on D which is analytic and self-adjoint. Suppose that $A(a), A(b)$ are invertible and that $A(z)$ satisfies the Virozub-Matsaev condition on Δ. Denote

$$Q(\Delta) = (2\pi i)^{-1} \oint_{\gamma} A(z)^{-1} dz$$

where $\gamma(\subset D)$ is a smooth curve which surrounds Δ, and $\mathcal{H}(\Delta) = \text{ran } Q(\Delta)$.

Under these conditions $A(z)$ has a self-adjoint linearization Λ in a Hilbert space $\mathcal{F}$. We construct a special operator $S(\in \mathcal{L}(\mathcal{H}(\Delta)))$ which is similar to the linearization $\Lambda(\in \mathcal{L}(\mathcal{F}))$. Since S acts in a subspace of the originally given space $\mathcal{H}$, we call S the inner linearization of $A(z)$.

We study various properties of the inner linearization and their connection with the properties of the operator function $A(z)$. We prove, in particular, the following factorization theorem for the restriction of $A(z)$ to the subspace $\mathcal{H}(\Delta)$.

Theorem. There exists an analytic in D operator function $M(z)$ with values in $\mathcal{L}(\mathcal{H}(\Delta), \mathcal{H})$ such that

$$A(z)f = M(z)(S - z)f \ (z \in D, f \in \mathcal{H}(\Delta)).$$

For each $z \in D$ the operator $M(z)$ is injective, and its range (depending on z) is a closed subspace of $\mathcal{H}$.

The talk is based on a joint work with H. Langer and V. Matsaev.