Schur complements in Krein spaces

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Given a (bounded) semidefinite positive operator A acting on a Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, Anderson-Trapp [1] defined the Schur complement (or shorted operator) $A_{/\mathcal{S}}$ by

$$A_{/\mathcal{S}} = \max_{\leq}\{X \in L(\mathcal{H}) : 0 \leq X \leq A,\ R(A) \subseteq \mathcal{S}^{\perp}\},$$

where $\leq$ stands for the usual order in the real vector space of selfadjoint operators. Later, Pekarev [2] proved that

$$A_{/\mathcal{S}} = A^{1/2} P_{\mathcal{M}^{\perp}} A^{1/2},$$

where $P_{\mathcal{M}^{\perp}}$ is the orthogonal projection onto the subspace $\mathcal{M}^{\perp} = A^{-1/2}(\mathcal{S}^{\perp})$.

The aim of this talk is to present a generalization of Schur complements to Krein spaces, following the ideas used by Pekarev to establish the above formula. In order to do so, recall that every J-selfadjoint operator $A \in L(\mathcal{H})$ can be factorized as $A = DD^{\#}$ where $D \in L(\mathcal{K}, \mathcal{H})$ is injective and $\mathcal{K}$ is another Krein space. However, this factorization is not unique, see [3].

Let $(\mathcal{H}, [\ ,\ ])$ be a Krein space with fundamental symmetry J. Given a (bounded) J-selfadjoint operator $A \in L(\mathcal{H})$ with the unique factorization property and a (suitable) closed subspace $\mathcal{S}$ of $\mathcal{H}$, the Schur complement $A_{/[\mathcal{S}]}$ is defined by

$$A_{/[\mathcal{S}]} = D P_{\mathcal{M}^{[\perp]}//\mathcal{M}} D^{\#},$$

where $P_{\mathcal{M}^{[\perp]}//\mathcal{M}}$ is the J-selfadjoint projection onto $\mathcal{M}^{[\perp]} = D^{-1}(\mathcal{S}^{[\perp]})$. Then, the basic properties of $A_{/[\mathcal{S}]}$ are developed and different characterizations are given, most of them resembling those of the shorted of (bounded) positive operators on a Hilbert space.

The talk is based on a joint work with A. Maestripieri [4].

References

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