Finite dimensional Sturm Liouville vessels and their tau functions

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In this work there is developed a theory of finite dimensional vessels, defined in [M, MV1, MVc], which originates at the work of M. Livšic [Liv1] and has many common points with [BV]. A key property of a vessel [Liv2] is that its transfer function intertwines solutions of Linear Differential Equations (LDEs) with a spectral parameter. In a special case, by choosing parameters of the vessel one obtains that its transfer function intertwines solutions of the Sturm Liouville [S, L] differential equations

$$-\frac{d^2}{dx^2}y(x) + q(x)y(x) = \lambda y(x) \tag{1}$$

with the spectral parameter λ and the *potential* $q(x)$, which is very similar to Crum transformations [Crum]. In this work it is supposed that the potential is continuously differentiable and that all its derivatives tend to zero as $x \to \infty$.

Tau function τ arises as the determinant of an invertible matrix [JMU, N] (and in general is the determinant of a Fredholm operator). As a result there arises a differential ring $\boldsymbol{\mathcal{R}}_*$ generated by $\frac{\tau'}{\tau}, e^{\int q}$, and it is proved that at the generic case ("almost always") all relevant objects (transfer function, solutions of SL equation (1)) belong to this special differential ring. For a given choice of spectral parameters, one can also analyze the Galois differential group, corresponding to the potential $q(x)$ and a choice of the spectral value λ. Solutions are Liouvillian [PS] in this case, since they are pure exponents, multiplied by polynomials.

This work can be considered as a first step toward analyzing and constructing Lax Phillips scattering theory [Pov, Fa] for Sturm Liouville differential equations on a half axis $(0, \infty)$ with singularity at 0. On the other hand, there is developed a rich and interesting theory of Vessels which has connections to the notion of τ function, arising in non linear differential equations and to the Galois differential theory for LDEs [PS, K, H].

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