The inverse Problem in the quantum theory of scattering using theory of vessels

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In this talk I am going to present a theory of Sturm Liouville vessels (developed in the finite-dimensional case in [M2], and originated in [Li, P, MV1, MVc, M, BV]), applied to the study of the inverse Problem in the quantum theory of scattering [F, LxPh]. This theory is a special case of a more general theory of vessels [AMV]. In the classical case, under assumption $\int_0^\infty x|q(x)|dx < \infty$ on the *potential* $q(x)$, one is interested in solutions of the Sturm Liouville (SL) differential equation [L, S]

$$-\frac{d^2}{dx^2}y(x) + q(x)y(x) = \lambda y(x), \quad \lambda \in \mathbb{C} \tag{1}$$

and compares them to the pure exponents, which can be viewed as solutions of the trivial SL equation, corresponding to $q(x) = 0$.

Transfer function of a Sturm Liouville vessel is of the form

$$S(x, \lambda) = I_2 - B^*(x)\mathbb{X}^{-1}(x)(\lambda I - A)^{-1}B(x)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $A, \mathbb{X}(x)$ are bounded operators on a Hilbert space $\mathcal{H}$, $B(x) : \mathcal{H} \to \mathbb{C}^2$ is bounded. Operators depends continuously on x and satisfy vessel conditions, which are linear differential equations. For a fixed $x = x_0$, the function $S(x, \lambda)$ is a realization of a Schur class function and Schur algorithm arises in the study of this problem, which actually corresponds to finite dimensional vessels [M2]. There also arises an interesting connection to Crum transformations [Crum] in that case.

We will construct Jost solutions and study the role of the tau function $\tau(x) = \det \mathbb{X}(x)$, where $\mathbb{X}(x) = I + T(X)$, for a trace class operator $T(x)$. Moreover, functions $\Omega(x, y), K(x, y)$ satisfying Gelfan-Levitan-Marchenko equation [F, Mar] will be constructed using the vessel.

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