Minimal spectral functions of an ordinary differential operator

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Let $l[y]$ be a formally selfadjoint differential expression of an even order $2n$ on the semiaxis $[0,\infty)$, let L_0 and $L(= L_0^*)$ be the corresponding minimal and maximal operator in $\mathfrak{H} = L_2[0,\infty)$ and let $\mathcal{D}$ by the domain of L.

Assume that $\hat{\mathcal{K}}$ is a subspace in $\mathbb{C}^{2n}$ and $(N_0 \;\; N_1) : \mathbb{C}^n \oplus \mathbb{C}^n \to \hat{\mathcal{K}}$ is a symmetric operator pair such that $\mathrm{ran}(N_0 \;\; N_1) = \hat{\mathcal{K}}$. By using a decomposing boundary triplet for L [1] we consider the boundary value problem

$$l[y] - \lambda y = f \tag{1}$$

$$N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + C'_{01}(\lambda)\Gamma'_0 y - C'_{11}(\lambda)\Gamma'_1 y = 0 \tag{2}$$

$$C'_{02}(\lambda)\Gamma'_0 y - C'_{12}(\lambda)\Gamma'_1 y = 0 \tag{3}$$

where $f \in \mathfrak{H}$, $y^{(1)}(0) = \{y^{[k-1]}(0)\}_{k=1}^n$ and $y^{(2)}(0) = \{y^{[2n-k]}(0)\}_{k=1}^n$ are vectors of quasi-derivatives at 0 and $\Gamma'_0 y$, $\Gamma'_1 y$ are boundary values of a function $y \in \mathcal{D}$ at the point $b = \infty$. Moreover we suppose that the matrices

$$C_0(\lambda) = \begin{pmatrix} N_0 & C'_{01}(\lambda) \\ 0 & C'_{02}(\lambda) \end{pmatrix}, \quad C_1(\lambda) = \begin{pmatrix} N_1 & C'_{11}(\lambda) \\ 0 & C'_{12}(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{4}$$

form a Nevanlinna pair $\mathcal{P} = (C_0(\lambda) : C_1(\lambda))$.

The problem (1)-(3) is a particular case of a general Nevanlinna boundary problem [2] and hence it generates a generalized resolvent and the corresponding spectral function $F_{\mathcal{P}}(t)$ of the operator L_0. In turn the problem (1)-(3) involves a decomposing boundary problem (when $C'_{01}(\lambda) = C'_{11}(\lambda) = 0$).

With the boundary problem (1)-(3) we associate the m-function $m_{\mathcal{P}}(\lambda)$, which is a uniformly strict Nevanlinna function with values in $[\hat{\mathcal{K}}]$. This function is a generalization of the classical Titchmarsh-Weyl function known for decomposing boundary problems.

Let $\mathcal{K}$ be a subspace in $\mathbb{C}^{2n}$ and let $\varphi(\cdot,\lambda)(\in [\mathcal{K}, \mathbb{C}])$ be an operator solution of the equation $l[y]-\lambda y=0$ with the constant initial data $\varphi^{(j)}(0,\lambda)=S_j,\ j\in\{0,1\},\ \lambda\in\mathbb{C}$ and such that $\operatorname{Ker}\varphi(t,\lambda)=\{0\}$. As is known a distribution $\Sigma_{\mathcal{P}}(\cdot):\mathbb{R}\to[\mathcal{K}]$ is called a spectral function of the boundary problem (1)-(3) corresponding to the solution φ if for each finite function $f\in\mathfrak{H}$ the Fourier transform $g_f(s):=\int\limits_0^b\varphi^*(t,s)f(t)\,dt$ satisfies the equality

$$((F_{\mathcal{P}}(b)-F_{\mathcal{P}}(a))f,f)_{\mathfrak{H}}=\int\limits_{[a,b)}(d\Sigma_{\mathcal{P}}(s)\,g_f(s),g_f(s)),\quad [a,b)\subset\mathbb{R}.$$

The natural problem is a description of spectral functions $\Sigma_{\mathcal{P}}(\cdot)=\Sigma_{\mathcal{P},\min}(\cdot)$ with the minimally possible value of $\dim\mathcal{K}$ [3] (we denote this value by $d_{\min}$). The complete solution of this problem is given in the following theorem.

Theorem 1. 1) Each selfadjoint boundary value problem admits the representation (1)-(3) with a selfadjoint pair $\mathcal{P}=(C_0:C_1)$ given by (4).

2) Let $\mathcal{P}$ be a selfadjoint pair (4) and let (1)-(3) be the corresponding boundary problem. Moreover assume that $\varphi_N(\cdot,\lambda)(\in[\hat{\mathcal{K}},\mathbb{C}])$ is an operator solution of the equation $l[y]-\lambda y=0$ with $\varphi_N^{(1)}(0,\lambda)=-N_0^*,\ \varphi_N^{(2)}(0,\lambda)=N_1^*$, $\lambda\in\mathbb{C}$. Then: (i) $n\le\dim\hat{\mathcal{K}}\le k$, where k is the defect number of L_0;

(ii) there exists a spectral function $\Sigma_{\mathcal{P},N}(\cdot):\mathbb{R}\to[\hat{\mathcal{K}}]$ of the problem (1)-(3) corresponding to φ_N;

(iii) $d_{\min}=\dim\hat{\mathcal{K}}$ and all minimal spectral functions are given by $\Sigma_{\mathcal{P},\min}(s)=X^*\Sigma_{\mathcal{P},N}(s)X$, where X is an isomorphism in $\hat{\mathcal{K}}$.

Corollary 2. Let under conditions of Theorem 1 $\widetilde{A}$ be a selfadjoint extension of L_0 defined by the boundary conditions (2), (3). Then the spectral multiplicity of the operator $\widetilde{A}$ does not exceed $\dim\hat{\mathcal{K}}(=\operatorname{rank}(N_0\ \ N_1))$.

Moreover for a fixed symmetric pair $(N_0\ \ N_1)$ we describe all spectral functions $\Sigma_{\mathcal{P},N}(s)$ in terms of the Nevanlinna boundary parameter $\mathcal{P}=(C_0(\lambda):C_1(\lambda))$. Such a description is given by means of the formula for m-functions $m_{\mathcal{P}}(\lambda)$ similar to the well known Krein formula for resolvents.

The above results can be extended to differential expressions with operator valued coefficients and arbitrary (possibly unequal) defect numbers.

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2. V.I. Mogilevskii // arXive:0909.3734v1 [math.FA] 21 Sep 2009.

3. N. Dunford and J.T. Schwartz. Linear operators. Part2. Spectral theory. New York, London: Interscience Publishers, 1963.