On the shift semigroup on the Hardy space of Dirichlet series

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We consider the Hardy space $\mathcal{H}^2$ of Dirichlet series

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}, \quad \Re(s) > 1/2,$$

with finite norm

$$\|f\|_{\mathcal{H}^2}^2 = \sum_{n=1}^{+\infty} |a_n|^2 < +\infty.$$

The space $\mathcal{H}^2$ was introduced by Hedenmalm, Lindqvist and Seip in their in 1997 paper as a Dirichlet series counterpart of the standard Hardy space of the unit disc.

For every positive integer $n \in \mathbb{Z}^+$ we have a natural operator $S(n)$ acting on $\mathcal{H}2$ given by multiplication by the Dirichlet monomial n^{-s}, that is,

$$S(n)f(s) = n^{-s} f(s), \quad \Re(s) > 1/2,$$

for $f \in \mathcal{H}^2$. This provides us with a function $S : \mathbb{Z}^+ \ni n \mapsto S(n)$ which is easily seen to be a multiplicative semigroup of isometries. We characterize this shift semigroup $S : \mathbb{Z}^+ \to \mathcal{L}(\mathcal{H}^2)$ up to unitary equivalence by means of a Wold decomposition. As an application we have that a shift invariant subspace of $\mathcal{H}^2$ is unitarily equivalent to $\mathcal{H}^2$ if and only if it has the form $\varphi\mathcal{H}^2$ for some $\mathcal{H}^2$-inner function φ.