Exploiting the natural block-structure of hierarchically-decomposed variational discretizations of elliptic boundary value problems

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Finite element discretizations of elliptic boundary value problems lead to large, sparse, but very ill-conditioned linear systems, which must be solved to within a certain tolerance to attain the optimal approximation quality of the computed finite element solution. Given a simplicial partition of the domain, the standard Lagrange space $\bar{V}$, consisting of globally-continuous functions which are polynomials of degree p when restricted to any simplex, is readily decomposed as $\bar{V} = \hat{V} \oplus \tilde{V}$, where $\hat{V}$ consists of piecewise-linear functions, and $\tilde{V}$ consists of the piecewise-degree-p-polynomials which vanish on the simplex vertices. This decomposition gives rise to a natural block-structure, with properties which can be exploited computationally by recognizing that:

1. the ill-conditioning of the matrix is the "fault" of the (relatively small) $(\hat{V}, \hat{V})$ block,
2. the (much larger) $(\tilde{V}, \tilde{V})$ block is spectrally equivalent to its diagonal,
3. a strong Cauchy inequality exists between the spaces $\hat{V}$ and $\tilde{V}$.

These considerations suggest a block Gauß-Seidel iteration which can be used either as a static iteration or as a preconditioner for a Krylov iteration (e.g. GMRES). After revisiting some of the known, but perhaps not well-known-enough results, we will provide a (hopefully new) argument that the full system is nearly solved to the required accuracy in the first Gauß-Seidel sweep. In the spirit of the aims of the minisymposium, and because the arguments seem most natural in this way, we will shift back and forth between finite element/variational and linear algebra notation and terminology.