Equiconvergence theorems for Sturm–Liouville operators with singular potentials

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We deal with the Sturm–Liouville operator

$$Ly = l(y) = -\frac{d^2y}{dx^2} + q(x)y,$$

with Dirichlet boundary conditions $y(0) = y(\pi) = 0$ in the space $L_2[0,\pi]$. We assume that the potential q is complex-valued and has the form $q(x) = u'(x)$, where $u \in W_2^\theta[0,\pi]$ with $0 < \theta < 1/2$. Here the derivative is treated in the distributional sense, and $W_2^\theta[0,\pi] = [L_2, W_2^1]_\theta$ is the Sobolev space with fractional order of smoothness defined by interpolation. We consider the problem of equiconvergence in $W_2^\theta[0,\pi]$ and $C^\theta[0,\pi]$–norm of two expansion of a function $f \in L_2[0,\pi]$. The first one is constructed using the system of the eigenfunctions and associated functions of the operator L, while the second one is the Fourier expansion in the series of sines.

Theorem. *Let $R > 0$, $0 < \theta < 1/2$. Consider operator L acting in the space $L_2[0,\pi]$ with the Dirichlet boundary conditions. Suppose that the complex-valued potential $q(x) = u'(x)$, where $u(x) \in B_{\theta,R}$.*

Let $\{y_n(x)\}_{n=1}^\infty$ be the system of the eigenfunctions and associated functions of the operator L and $\{w_n(x)\}_{n=1}^\infty$ be the biorthogonal system.

For an arbitrary function $f \in L_2[0,\pi]$ denote

$$c_n := (f(x), w_n(x)), \quad c_{n,0} := \sqrt{2/\pi}(f(x), \sin nx).$$

Then

1) there exist a natural number $M = M_{\theta,R}$ and a positive number $C = C_{\theta,R}$ such that for all $m \geq M$ and for all $f \in L_2[0,\pi]$

$$\left\| \sum_{n=1}^m c_n y_n(x) - \sum_{n=1}^m \sqrt{\frac{2}{\pi}} c_{n,0} \sin nx \right\|_{W_2^\theta} \leq C \left(\sqrt{\sum_{n \geq m^{1-\theta}} |c_{n,0}|^2} + \frac{\|f\|_{L_2}}{m^{\theta(1-\theta)}} \right).$$

2)

$$\left\| \sum_{n=1}^{m} c_n y_n(x) - \sum_{n=1}^{m} \sqrt{\frac{2}{\pi}} c_{n,0} \sin nx \right\|_{C^\theta} \to 0, \quad m \to +\infty.$$