THE MATHEMATICAL WORK OF JÜRGEN GÄRTNER

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§ Legacy

Over the past 35 years, Jürgen has made seminal contributions to probability theory and analysis. In this laudatio, I will describe what I consider to be his five most important works:

(1) Gärtner-Ellis LDP.
(2) KPP-equation.
(3) Dawson-Gärtner projective limit LDP.
(4) McKean-Vlasov equation.
(5) Parabolic Anderson Model.

What characterizes the papers of Jürgen is that they all deal with fundamental and hard problems, which for their solution require a delicate combination of techniques from probability theory and analysis.

A red thread through his work is the symbiosis of large deviation theory and potential theory, which he masterfully combines to reach powerful and elegant solutions.

(1) Gärtner-Ellis LDP

In 1977, Jürgen proved what is nowadays considered to be the most general form of Cramér's theorem in LD-theory. This line of work, which was suggested to him by Mark Freidlin, took place while LD-theory itself was under construction.

> As such, Jürgen's theorem belongs to the very heart of LD-theory, as developed in the 1970's by Freidlin & Wentzell and Donsker & Varadhan.

In 1984, the assumptions under which Jürgen had proved his theorem were weakened by Richard Ellis.

THEOREM: *Given a sequence* $(X_n)_{n\in\mathbb{N}}$ *of random variables taking values in* $\mathbb{R}^d$*, let*

$$\phi_n(t) = E(e^{(t,X_n)}), \qquad t \in \mathbb{R}^d,$$

denote their moment generating functions. Suppose that

$$\lim_{n\to\infty} \frac{1}{n} \log \phi_n(nt) = \Phi(t) \text{ exists for all } t \in \mathbb{R}^d,$$

and is everywhere finite and differentiable. Then the family $(P_n)_{n\in\mathbb{N}}$ *with* $P_n(\cdot) = P(X_n \in \cdot)$ *satisfies the LDP with rate function* $I\colon \mathbb{R}^d \to [0,\infty]$ *given by*

$$I(x) = \sup_{t\in\mathbb{R}^d} [(t,x) - \Phi(t)], \qquad x \in \mathbb{R}^d.$$

Informally, the LDP says that

$$P(X_n \approx x) \approx e^{-nI(x)}, \qquad x \in \mathbb{R}^d,\ n \to \infty,$$

and therefore gives full control of the fluctuations of the random variable X_n for large n.

For the special case where $X_n = \frac{1}{n}(Y_1 + \cdots + Y_n)$, $n \in \mathbb{N}$, with $(Y_i)_{i\in\mathbb{N}}$ i.i.d., this theorem reduces to Cramér's theorem for the empirical mean. However, in its full generality, the theorem is applicable way beyond the i.i.d. setting, including Markov processes, Gibbs random fields, and random processes in random media.

Over the years, the Gärtner-Ellis LDP has become one of the workhorses of LD-theory!

(2) KPP-equation

In 1982, Jürgen wrote a seminal paper on the famous semi-linear diffusion equation introduced by Kolmogorov, Petrovskii & Piskunov in 1937:

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta u(x,t) + f(u(x,t)), \quad x \in \mathbb{R}^d,\ t \geq 0.$$

Here, $f\colon [0,1] \to [0,\infty)$ is assumed to be once continuously differentiable with $f(0) = f(1) = 0$ and $0 < f(u)/u \leq f'(0)$ for $u \in (0,1)$. The initial condition is taken to be

$$u(x,0) = g(x), \qquad x \in \mathbb{R}^d,$$

for some appropriate $g\colon \mathbb{R}^d \to [0,1]$ that is strictly positive near $x = 0$ and tends to zero rapidly at infinity.

THEOREM: *Abbreviate* $v^* = [2f'(0)]^{1/2}$*, and define* $h(z) = \sup_{u\in(0,z]}[f'(0) - f(u)/u]$ *for* $z \in (0,1]$*. Suppose that*

$$\int_0^1 dz\, h(z)\, z^{-1} \log^2(1/z) < \infty,$$

and that $g(x) = \bar{g}(\|x\|)$ *with*

$$\limsup_{r\to\infty} r^{-1/2} \log[e^{v^* r}\bar{g}(r)] < \infty.$$

Then, for every $\epsilon \in (0, \frac{1}{2})$*, there exists a* $\rho(\epsilon) \in (0,\infty)$ *such that, for all* t *sufficiently large,*

$$\Big\{x \in \mathbb{R}^d\colon\ \epsilon < u(x,t) < 1-\epsilon\Big\}$$
$$\subset \Big\{x \in \mathbb{R}^d\colon\ m(t) - \rho(\epsilon) < \|x\| < m(t) + \rho(\epsilon)\Big\},$$

where

$$m(t) = v^* t - \frac{d+2}{2v^*} \log t + \frac{1}{v^*} \log \int_0^\infty dr\, e^{-r^2/2t}\, r^{(d+1)/2}\, e^{v^* r}\, \bar{g}(r).$$

This result identifies the location of the expanding wave front around which u drops from $u \approx 1$ to $u \approx 0$.

Earlier work by McKean (1975–1976), Aronson & Weinberger (1978), Bramson (1978) and Uchiyama (1978) had failed to identify the constant and had required much more severe restrictions on g, such as compact support, or $d = 1$.

The proof centers around a very delicate estimate of the first-exit time distribution for a Brownian motion in a time-dependent domain. In later work, Jürgen extended his result to a much broader class of reaction-diffusion equations.

(3) Dawson-Gärtner projective limit LDP

In 1987, Jürgen and Don Dawson proved a theorem that takes a nested sequence of LDP's and uses this to obtain a new LDP via a projective limit.

This theorem is a powerful tool, because it allows us to first derive an LDP in a simple setting (e.g. on a finite or compact state space) and then draw from that an LDP in a more difficult setting (e.g. on an infinite or non-compact state space).

Over the years, also the Dawson-Gärtner projective limit LDP has become one of the workhorses of LD-theory!

THEOREM: *Let* $(P_n)_{n\in\mathbb{N}}$ *be a family of probability measures on a Hausdorff topological space* χ. *Let* $(\pi^N)_{N\in\mathbb{N}}$ *be a nested family of projections acting on* χ, *and put*

$$\chi^N = \pi^N \chi, \qquad P_n^N = P_n \circ (\pi^N)^{-1},$$

If, for each $N \in \mathbb{N}$, *the family* $(P_n^N)_{n\in\mathbb{N}}$ *satisfies the LDP on* χ^N *with rate function* $I^N \colon \chi^N \to [0,\infty]$, *then also the family* $(P_n)_{n\in\mathbb{N}}$ *satisfies the LDP on* χ *with rate function* $I \colon \chi \to [0,\infty]$ *given by*

$$I(x) = \sup_{N\in\mathbb{N}} I^N(\pi^N x), \qquad x \in \chi.$$

(4) McKean-Vlasov equation

In 1987–1989, Jürgen and Don Dawson wrote a series of papers on the McKean-Vlasov equation.

Let $H_N\colon \mathbb{R}^N \to \mathbb{R}$ be the N-particle mean-field Hamiltonian

$$H_N(x) = \frac{1}{2N} \sum_{i,j=1}^{N} f(x_j - x_i) + \sum_{i=1}^{N} g(x_i), \qquad x = (x_1, \ldots, x_N),$$

with f even and f, g twice continuously differentiable. For $T > 0$, let $(X(t))_{t \in [0,T]}$ evolve according to the system of N coupled diffusion equations

$$dX_i(t) = \frac{\partial H_N}{\partial x_i}(X(t))\, dt + dB_i(t), \qquad i = 1, \ldots, N,$$

where $(B_i(t))_{t\in[0,T]}$, $i = 1, \dots, N$, are i.i.d. Brownian motions. Define the empirical path measure

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_i(t))_{t\in[0,T]}}.$$

THEOREM: *The family* $(P_N)_{N\in\mathbb{N}}$ *with* $P_N(\cdot) = P(L_N \in \cdot)$ *satisfies the LDP with rate function* $I\colon M_1(C[0,T]) \to [0,\infty]$ *given by*

$$I(Q) = \int dQ \log\left(\frac{dQ}{dP^Q}\right),$$

where P^Q *is the law of a single diffusion with a certain self-interaction (as is typical for mean-field models).*

The rate function I has a unique zero solving the equation

$$Q = P^Q,$$

the solution of which is given by the so-called McKean-Vlasov equation. In terms of the latter, I can be written as an action functional, in the spirit of Freidlin-Wentzell-theory.

Related work was done by Sznitman (1984) and Ben Arous & Brunaud (1990).

The results were later extended to random mean-field interactions by Dai Pra & den Hollander (1996) and to spin-glass mean-field interactions by Ben Arous & Guionnet (1990–1997) and Jürgen's student Grunwald (1996).

In 1991-1997, while extending their work on the McKean-Vlasov equation, Jürgen and Don Dawson introduced the notion of multi-level large deviations, describing the LD-behavior of multi-array families of dependent random variables.

This work in turn gave rise to the Dawson-Greven renormalization program for hierarchically interacting diffusions, introduced in 1994 and since then pursued by various groups.

(5) Parabolic Anderson Model

In 1990, Jürgen wrote a seminal paper with Stas Molchanov on intermittency in the Parabolic Anderson Model.

> A lot of earlier work had been done in the physics and chemistry literature, but this was the first paper that put the model on a firm mathematical basis and provided a new way to look at intermittency via the study of Lyapunov exponents.

A follow-up paper in 1998 pushed the subject further. Since then Jürgen has been working intensively on the PAM with several colleagues, both senior and junior. There are two versions:

- static
- dynamic

The PAM is the partial differential equation:

$$\frac{\partial u}{\partial t}(x,t) = \kappa \Delta u(x,t) + \xi(x,t)\, u(x,t), \quad x \in \mathbb{Z}^d,\ t \geq 0,$$

where Δ is the discrete Laplacian, κ is the diffusion constant and $\xi(x,t)$ is a space-time random medium that drives the equation. Typical initial conditions are:

$$u(x,0) = 1 \quad \text{or} \quad u(x,0) = \delta_0(x).$$

The key objects of interest are the Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \frac{1}{pt} \log E([u(0,t)]^p), \quad p > 0,$$
$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log u(0,t), \quad \xi - a.s.,$$

where E denotes expectation over the ξ-field. The PAM is said to be intermittent when

$$p \mapsto \lambda_p \text{ is strictly increasing.}$$

> The geometric interpretation behind this property is that the u-field develops sparse high peaks, with λ_p being dominated by different classes of peaks for different p.

- STATIC

For the case where ξ is time-independent, the PAM is by now fairly well-understood. The typical case is where $\xi(x)$, $x \in \mathbb{Z}^d$, are i.i.d., and there are four subclasses of distributions of $\xi(0)$ leading to qualitatively different behavior.

A detailed description has been obtained for the location and the height of the dominant peaks in the u-field, which tend to concentrate around the peaks in the ξ-field.

• DYNAMIC

For the case where ξ is time-dependent, work is still in progress. Early work was done by R. Carmona and S. Molchanov when ξ consists of i.i.d. Brownian noises.

Since then the focus has been on a number of choices where ξ evolves like an interacting particle system:

(1) Independent random walks.
(2) Exclusion process.
(3) Voter model.

It turns out that the behavior of λ_p as a function of d and κ is extremely rich and challenging. Much remains to be clarified.

The development of the PAM took place parallel to the work by Alain Sznitman on Brownian motion among Poissonian obstacles. Both areas have substantially enriched our understanding of random processes in random media.

The main collaborators of Jürgen on the PAM have been
S. Molchanov, F. den Hollander, W. König, G. Maillard.

Many others have made important contributions, including
M. Biskup, M. Cranston, R. van der Hofstad, H. Kesten, H.-Y. Kim, L. Koralov, P. Mörters, T. Mountford, T. Shiga, V. Sidoravicius, N. Sidorova, F. Viens, A. Vizcarra,
and Jürgen's students
A. Drewitz, J. Hähnel, M. Heydenreich, A. Schnitzler, T. Wolff.

§ Three inspirators

Over the years, three collaborators of Jürgen have been a major inspiration to him:

- Mark Freidlin
- Don Dawson
- Stas Molchanov

Each of them has played an important role in his career, and has drawn him into exciting new areas of research, which he has subsequently pursued with all his force.

§ Personal remarks

For me, personally, it has been a wonderful experience to work with Jürgen. We wrote 7 papers together, and number 8 is in progress. Our discussions over the past 20 years have covered an enormous area, and have been both fruitful and enjoyable.

What I value most in Jürgen, apart from his mastery of probability theory and analysis, is his ability to look far ahead, his constant search for elegance, his unwavering computational skills, his humour and scepticism, as well as his friendship and loyalty.

Jürgen holds the record as the most frequent visitor at EURANDOM. I trust that he will continue to push up this record in the years to come!

Happy birthday Jürgen!

We all hope you will enjoy this workshop,
for which we have gathered here in your honor.