Nonlinear Model Reduction via Quadratic Bilinear Control Systems

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Computational Methods in Systems and Control Theory
Outline

1. Nonlinear Model Order Reduction
2. Multimoment-Matching for QBDAEs
3. $\mathcal{H}_2$-Model Reduction for Bilinear Systems
4. Outlook
Motivation

Given a large-scale state-nonlinear control system of the form

\[
Σ : \begin{cases}
  \dot{x}(t) = f(x(t)) + bu(t), \\
y(t) = cx(t), \\
x(0) = x_0,
\end{cases}
\]

with \( f : \mathbb{R}^n \to \mathbb{R}^n \) nonlinear and \( b, c^T \in \mathbb{R}^n, x \in \mathbb{R}^n, u, y \in \mathbb{R} \).
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\[ \xrightarrow{\text{MOR}} \]

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with $\hat{f} : \mathbb{R}^\hat{n} \to \mathbb{R}^\hat{n}$ and $\hat{b}, \hat{c}^T \in \mathbb{R}^\hat{n}$, $x \in \mathbb{R}^\hat{n}$, $u \in \mathbb{R}$ and $\hat{y} \approx y \in \mathbb{R}$, $\hat{n} \ll n$. 

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T. Breiten, Nonlinear Model Reduction via Quadratic Bilinear Control Systems
Common Reduction Techniques

Proper Orthogonal Decomposition (POD)

- Take computed or experimental ’snapshots’ of full model:
  \[ [x(t_1), x(t_2), \ldots, x(t_N)] =: X, \]
- perform SVD of snapshot matrix: \( X = VSW^T \approx V_n S_n W_n^T. \)
- Reduction by POD-Galerkin projection: \( \dot{x} = V_n^T f(V_n \hat{x}) + V_n^T Bu. \)
- Requires evaluation of \( f \)
  \[ \leadsto \text{discrete empirical interpolation [Sorensen/Chaturantabut '09].} \]
- Input dependency due to ’snapshots’!
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Trajectory Piecewise Linear (TPWL)

- Linearize \( f \) along trajectory,
- reduce resulting linear systems,
- construct reduced model by weighting sum of linear systems.
- Requires simulation of original model and several linear reduction steps, many heuristics.
State-Space Representation of QBDAEs

We will consider quadratic-bilinear SISO systems of the form

\[
\begin{align*}
E \dot{x} &= A_1 x + A_2 x \otimes x + N x u + b u \\
y &= c^T x
\end{align*}
\]

where \( E, A_1, N \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times n^2} \) (Hessian tensor), \( b, c^T \in \mathbb{R}^n \).

- A large class of nonlinear control-affine systems can be transformed into the above type of control system.
- The transformation is exact, but a slight increase of the state dimension has to be accepted.
- Input-output behavior can be characterized by generalized transfer functions \( \leadsto \) enables us to use Krylov-based reduction techniques.
Nonlinear Model Order Reduction Multimoment-Matching for QBDAEs

Transformation via McCormick Relaxation

**Theorem [Gu’09]**

Assume that the state equation of a nonlinear system $\Sigma$ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, $\Sigma$ can be transformed into a system of quadratic-bilinear DAEs of dimension $N > n$. 
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Example

- $\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}, \quad \dot{x}_2 = -x_2 + u$. 
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Variational Analysis and Linear Subsystems

Analysis of nonlinear systems by variational equation approach:

\[
\begin{align*}
\alpha u(t),
\text{nonlinear system is assumed to be a series of homogeneous nonlinear subsystems, i.e. response should be of the form} \\
x(t) &= \alpha x_1(t) + \alpha_2 x_2(t) + \alpha_3 x_3(t) + \ldots \\
\text{comparison of terms } \alpha_i, i = 1, 2, \ldots \text{leads to series of systems} \\
E \dot{x}_1 &= A_1 x_1 + B u \\
E \dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + N x_1 u \\
E \dot{x}_3 &= A_1 x_3 + A_2 (x_1 \otimes x_2 + x_2 \otimes x_1) + N x_2 u \\
\text{although } i\text{-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms } x_j, j < i, \text{are interpreted as pseudo-inputs.}
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Generalized Transfer Functions

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H_1(s_1) = C (s_1 E - A_1)^{-1} B, \\
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\[
H_2(s_1, s_2) = \frac{1}{2!} C ((s_1 + s_2) E - A_1)^{-1} [N (G_1(s_1) + G_1(s_2)) \\
+ A_2 (G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1))] ,
\]

\[
H_3(s_1, s_2, s_3) = \frac{1}{3!} C ((s_1 + s_2 + s_3) E - A_1)^{-1} [N (G_1(s_1) + G_1(s_2)) \\
+ A_2 (G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)) + A_3 (G_1(s_1) \otimes G_1(s_2) \otimes G_1(s_3) + G_1(s_2) \otimes G_1(s_3) \otimes G_1(s_1) + G_1(s_3) \otimes G_1(s_1) \otimes G_1(s_2))] ,
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Characterization via Multimoments

For simplicity, focus on the first two transfer functions. For $H_1(s_1)$, choosing $\sigma$ and making use of the Neumann lemma leads to

$$H_1(s_1) = \sum_{i=0}^{\infty} C \{ (A_1 - \sigma E)^{-1} E \}^i (A_1 - \sigma E)^{-1} B (s_1 - \sigma)^i m_{s_1, \sigma}^i.$$
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Similarly, specifying an expansion point $(\tau, \xi)$ yields

$$H_2(s_1, s_2) = \frac{1}{2} \sum_{i=0}^{\infty} \left( (A_1 - (\tau + \xi) E)^{-1} E \right)^i \left( (A_1 - (\tau + \xi) E)^{-1} (s_1 + s_2 - \tau - \xi) \right)^i.$$

$$[A_2 \left( \sum_{j=0}^{\infty} m^j_{s_1, \tau} \otimes \sum_{k=0}^{\infty} m^k_{s_2, \xi} + \sum_{k=0}^{\infty} m^k_{s_2, \xi} \otimes \sum_{j=0}^{\infty} m^j_{s_1, \tau} \right) + N \left( \sum_{p=0}^{\infty} m^p_{s_1, \tau} + \sum_{p=0}^{\infty} m^q_{s_2, \xi} \right)]$$
Constructing the Projection Matrix

For derivatives around $\sigma = \tau = \xi$ up to order $q - 1$, construct the Krylov spaces:

\begin{align*}
U_i &= K_q \left( (A_1 - \sigma E)^{-1} E, (A_1 - \sigma E)^{-1} B \right) \\
W_i &= K_{q-i+1} \left( (A_1 - 2\sigma E)^{-1} E, (A_1 - 2\sigma E)^{-1} N U_i \right) \\
Z_i &= K_{q-i-j+2} \left( (A_1 - 2\sigma E)^{-1} E, (A_1 - 2\sigma E)^{-1} A_2 (U_i \otimes U_j + U_j \otimes U_i) \right)
\end{align*}

$U_i$ denoting the $i$-th column of $U$.

Set $V = \text{orth}(\left[ U, W, Z \right])$ and construct $\hat{\Sigma}$ by the Galerkin-Projection $P = V V^T$:

\begin{align*}
\hat{A}_1 &= V^T A_1 V \in \mathbb{R}^{\hat{n} \times \hat{n}} \\
\hat{A}_2 &= V^T A_2 V \otimes V \in \mathbb{R}^{\hat{n} \times \hat{n}^2} \\
\hat{N} &= V^T N V \in \mathbb{R}^{\hat{n} \times \hat{n}} \\
\hat{B} &= V^T B \in \mathbb{R}^{\hat{n}} \\
\hat{C}^T &= V^T C \in \mathbb{R}^{\hat{n}}
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$$U = \mathcal{K}_q \left( (A_1 - \sigma E)^{-1} E, (A_1 - \sigma E)^{-1} B \right)$$
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$$W_i = \mathcal{K}_{q-i+1} \left( (A_1 - 2\sigma E)^{-1} E, (A_1 - 2\sigma E)^{-1} NU_i \right),$$
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for $j = 1 : \min(q - i + 1, i)$

$$Z_i = \mathcal{K}_{q-i-j+2} \left((A_1 - 2\sigma E)^{-1} E, \right.$$ 
$$\left. (A_1 - 2\sigma E)^{-1} A_2 (U_i \otimes U_j + U_j \otimes U_i)\right),$$

$U_i$ denoting the $i$-th column of $U$. 
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$U_i$ denoting the $i$-th column of $U$. Set $V = \text{orth}([U, W, Z])$ and construct $\hat{\Sigma}$ by the Galerkin-Projection $\mathcal{P} = VV^T$:

$$\hat{A}_1 = V^T A_1 V \in \mathbb{R}^{\hat{n} \times \hat{n}} , \quad \hat{A}_2 = V^T A_2 V \otimes V \in \mathbb{R}^{\hat{n} \times \hat{n}^2} ,$$

$$\hat{N} = V^T N V \in \mathbb{R}^{\hat{n} \times \hat{n}} , \quad \hat{B} = V^T B \in \mathbb{R}^{\hat{n}} , \quad \hat{C}^T = V^T C \in \mathbb{R}^{\hat{n}} .$$
The FitzHugh-Nagumo System

- FitzHugh-Nagumo system modeling a neuron

\[ εv_t(x, t) = ε^2 v_{xx}(x, t) + f(v(x, t)) - w(x, t) + g, \]
\[ w_t(x, t) = hv(x, t) - γw(x, t) + g, \]

with \( f(v) = v(v - 0.1)(1 - v) \) and initial and boundary conditions

\[
\begin{align*}
v(x, 0) &= 0, & w(x, 0) &= 0, & x \in [0, 1], \\
v_x(0, t) &= -i_0(t), & v_x(1, t) &= 0, & t \geq 0,
\end{align*}
\]

where
\[
ε = 0.015, \quad h = 0.5, \quad γ = 2, \quad g = 0.05, \quad i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)
\]

- original state dimension \( n = 2 \cdot 400 \), QBDAE dimension \( N = 3 \cdot 400 \),
- reduced QBDAE dimension \( r = 26 \), chosen expansion point \( σ = 1 \)
- 3D phase space
Next, let us focus on a nonlinear PDE arising in jet diffusion flame models

\[
\frac{\partial w}{\partial t} + U \cdot \nabla w - \nabla (\kappa \nabla w) + f(w) = 0, \quad (x, t) \in (0, 1) \times (0, T),
\]

with Arrhenius type term \( f(w) = Aw(c - w)e^{-\frac{E}{d-w}} \) and constant parameters \( U, A, E, c, d, \kappa \). Again define initial and boundary conditions:

\[
\begin{align*}
    w(x, 0) &= 0, \quad x \in [0, 1], \\
    w(0, t) &= u(t), \quad t \geq 0, \\
    w(1, t) &= 0, \quad t \geq 0, \\
    w_{\text{center}} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.
\end{align*}
\]

Figure: [Kurose]
Jet Diffusion Flame Model

Transient responses for $k = 1500$ and $u(t) = \frac{1}{2} \cos\left(\frac{\pi t}{5} + 1\right)$
Jet Diffusion Flame Model

Relative errors for $k = 1500$ and $u(t) = \frac{1}{2} \cos\left(\frac{\pi t}{5} + 1\right)$

- $\sigma = -1, \hat{n} = 8$
- $\sigma = 1, \hat{n} = 8$
State-Space and Output Representation

Let us now focus on the special case of bilinear control systems:

\[ \Sigma : \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i x(t)u(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases} \]

where \( A, N_i \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}. \)
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\end{cases}
\]

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Output Characterization (SISO): Volterra series

\[
y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-1}} h(t_1, \ldots, t_k) u(t-t_1-\ldots-t_k) \cdot \cdots \cdot u(t-t_k) dt_i \cdots dt_1,
\]

with kernels \( h(t_1, \ldots, t_k) = Ce^{At_k} N \cdots e^{At_2} Ne^{At_1} B. \)
State-Space and Output Representation

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\]

with kernels \( h(t_1, \ldots, t_k) = Ce^{A t_k} N \cdots e^{A t_2} Ne^{A t_1} B. \)

**Multivariable Laplace-transform (SISO):**

\[
H_i(s_1, \ldots, s_i) = C(s_i I - A)^{-1} N \cdots (s_2 I - A)^{-1} N(s_1 I - A)^{-1} B.
\]
$\mathcal{H}_2$-Norm for Bilinear Systems

One possible generalization of the known linear $\mathcal{H}_2$-norm is given by:

$$||\Sigma||^2_{\mathcal{H}_2} := \text{tr} \left( \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} H_k(i\omega_1, \ldots, i\omega_k) H_k^T(i\omega_1, \ldots, i\omega_k) \right),$$

where $H_k$ denotes the $k$-th transfer function associated with the bilinear system.
One possible generalization of the known linear $\mathcal{H}_2$-norm is given by:

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where $H_k$ denotes the $k$-th transfer function associated with the bilinear system.

It can be shown that we can alternatively compute

$$||\Sigma||^2_{\mathcal{H}_2} = (\text{vec}(I_p))^T (C \otimes C) \left( -A \otimes I - I \otimes A - \sum_{k=1}^{m} N_k \otimes N_k \right)^{-1} (B \otimes B) \text{vec}(I_m).$$
\( \mathcal{H}_2 \)-Norm for Bilinear Systems

One possible generalization of the known linear \( \mathcal{H}_2 \)-norm is given by:

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\[
||\Sigma||^2_{\mathcal{H}_2} = (\text{vec}(I_p))^T (C \otimes C) \left( -A \otimes I - I \otimes A - \sum_{k=1}^{m} N_k \otimes N_k \right)^{-1} (B \otimes B) \text{vec}(I_m).
\]

In order to find an \( \mathcal{H}_2 \)-optimal reduced system, we define the error system \( \Sigma^{err} \) as follows:

\[
A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad N_k^{err} = \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.
\]
Necessary $\mathcal{H}_2$-Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$\hat{A} = R\Lambda R^{-1}, \quad \hat{N}_k = R^{-1}\hat{N}_k R, \quad \hat{B} = R^{-1}\hat{B}, \quad \hat{C} = \hat{C} R.$$
Necessary $\mathcal{H}_2$-Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$
\hat{A} = R\Lambda R^{-1}, \quad \hat{\mathcal{N}}_k = R^{-1}\tilde{N}_k R, \quad \hat{B} = R^{-1}\tilde{B}, \quad \hat{C} = \tilde{C} R.
$$

Using $\Lambda$, $\hat{\mathcal{N}}_k$, $\hat{B}$, $\hat{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality:
Necessary $\mathcal{H}_2$-Optimality Conditions

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Using $\Lambda$, $\hat{N}_k$, $\hat{B}$, $\hat{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality:

$$(\text{vec}(I_p))^T \left( e_i e_j^T \otimes C \right) \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^m \hat{N}_k \otimes \hat{N}_k \right)^{-1} \left( \hat{B} \otimes B \right) \text{vec}(I_m)$$

$$= (\text{vec}(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^m \hat{N}_k \otimes \hat{N}_k \right)^{-1} \left( \hat{B} \otimes \hat{B} \right) \text{vec}(I_m).$$
Let us assume \( \hat{\Sigma} \) is given by its eigenvalue decomposition:

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\hat{A} = R\Lambda R^{-1}, \quad \hat{N}_k = R^{-1}\hat{N}_k R, \quad \hat{B} = R^{-1}\hat{B}, \quad \hat{C} = \hat{C} R.
\]

Using \( \Lambda, \hat{N}_k, \hat{B}, \hat{C} \) as optimization parameters, we can derive necessary conditions for \( \mathcal{H}_2 \)-optimality:

\[
(vec(I_p))^T \left( e_i e_j^T \otimes C \right) \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \hat{N}_k \otimes \hat{N}_k \right)^{-1} \left( \hat{B} \otimes B \right) vec(I_m)
\]

\[
= (vec(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \hat{N}_k \otimes \hat{N}_k \right)^{-1} \left( \hat{B} \otimes \hat{B} \right) vec(I_m).
\]

\[
(vec(I_p))^T \left( e_i e_j^T \otimes C \right) ( -\Lambda \otimes I_n - I_{\hat{n}} \otimes A)^{-1} \left( \hat{B} \otimes B \right) vec(I_m)
\]

\[
= (vec(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) ( -\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A})^{-1} \left( \hat{B} \otimes \hat{B} \right) vec(I_m).
\]
**Necessary $\mathcal{H}_2$-Optimality Conditions**

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

\[
\hat{A} = R\Lambda R^{-1}, \quad \hat{N}_k = R^{-1}\hat{N}_k R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{CR}.
\]

Using $\Lambda$, $\hat{N}_k$, $\tilde{B}$, $\tilde{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality:

\[
(vec(I_p))^T \left( e_i e_j^T \otimes C \right) \left( -\Lambda \otimes I_n - \hat{l}_\mathcal{h} \otimes A - \sum_{k=1}^{m} \hat{N}_k \otimes \hat{N}_k \right)^{-1} \left( \tilde{B} \otimes B \right) vec(l_m)
\]

\[
= (vec(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - \hat{l}_\mathcal{h} \otimes \hat{A} - \sum_{k=1}^{m} \hat{N}_k \otimes \hat{N}_k \right)^{-1} \left( \tilde{B} \otimes \hat{B} \right) vec(l_m).
\]

\[
(vec(I_p))^T \left( e_i e_j^T \otimes C \right) \left( -\Lambda \otimes I_n - \hat{l}_\mathcal{h} \otimes A \right)^{-1} vec(\hat{B}\hat{B}^T)
\]

\[
= (vec(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - \hat{l}_\mathcal{h} \otimes \hat{A} \right)^{-1} vec(\hat{B}\hat{B}^T).
\]
Necessary $\mathcal{H}_2$-Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$\hat{A} = R\Lambda R^{-1}, \quad \tilde{N}_k = R^{-1}\hat{N}_k R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{C} R.$$

Using $\Lambda$, $\tilde{N}_k$, $\tilde{B}$, $\tilde{C}$ as optimization parameters, we can derive necessary conditions for $\mathcal{H}_2$-optimality:

$$\begin{align*}
    (\text{vec}(I_p))^T \left( e_i e_j^T \otimes C \right) 
    \begin{pmatrix}
        -\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k \otimes \hat{N}_k
    \end{pmatrix}^{-1}
    \begin{pmatrix}
        \tilde{B} \otimes B
    \end{pmatrix}
    \text{vec}(I_m)
\end{align*}$$

$$= (\text{vec}(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) 
    \begin{pmatrix}
        -\lambda_1 I - A \\
        \ddots \\
        -\lambda_{\hat{n}} I - A
    \end{pmatrix}^{-1}
    \begin{pmatrix}
        B \tilde{B}_I^T \\
        \vdots \\
        B \tilde{B}_{\hat{n}}^T
    \end{pmatrix}$$

$$= (\text{vec}(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) 
    \begin{pmatrix}
        -\lambda_1 I - \hat{A} \\
        \ddots \\
        -\lambda_{\hat{n}} I - \hat{A}
    \end{pmatrix}^{-1}
    \begin{pmatrix}
        \hat{B} \tilde{B}_I^T \\
        \vdots \\
        \hat{B} \tilde{B}_{\hat{n}}^T
    \end{pmatrix}.$$
Let us assume \( \hat{\Sigma} \) is given by its eigenvalue decomposition:

\[
\hat{A} = R\Lambda R^{-1}, \quad \tilde{N}_k = R^{-1}\hat{N}_k R, \quad \hat{B} = R^{-1}\hat{B}, \quad \hat{C} = \hat{C} R.
\]

Using \( \Lambda, \tilde{N}_k, \hat{B}, \hat{C} \) as optimization parameters, we can derive necessary conditions for \( \mathcal{H}_2 \)-optimality:

\[
(\text{vec}(I_p))^T \left( e_i e_j^T \otimes C \right) \left( -\Lambda \otimes I_n - I_{\hat{h}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k \otimes N_k \right)^{-1} \left( \hat{B} \otimes B \right) \text{vec}(I_m)
\]

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= (\text{vec}(I_p))^T \left( e_i e_j^T \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - I_{\hat{h}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k \otimes \hat{N}_k \right)^{-1} \left( \hat{B} \otimes \hat{B} \right) \text{vec}(I_m).
\]

\[
\mathcal{H}(-\lambda_j)\tilde{B}_j^T = \hat{\mathcal{H}}(-\lambda_j)\hat{B}_j^T
\]
A Bilinear IRKA Approach

Algorithm 1 Bilinear IRKA

**Input:** $A, N_k, B, C, \hat{A}, \hat{N}_k, \hat{B}, \hat{C}$

**Output:** $A^{opt}, N_k^{opt}, B^{opt}, C^{opt}$

1: while (change in $\Lambda > \epsilon$) do
2: $R\Lambda R^{-1} = \hat{A}, \tilde{B} = R^{-1}\hat{B}, \tilde{C} = \hat{C}R, \tilde{N}_k = R^{-1}\hat{N}_k R$
3: $\text{vec}(V) = \left( -\Lambda \otimes I_n - I_{\tilde{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k \otimes N_k \right)^{-1} \left( \tilde{B} \otimes B \right) \text{vec}(I_m)$
4: $\text{vec}(W) = \left( -\Lambda \otimes I_n - I_{\tilde{n}} \otimes A^T - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k^T \right)^{-1} \left( \tilde{C}^T \otimes C^T \right) \text{vec}(I_p)$
5: $V = \text{orth}(V), W = \text{orth}(W)$
6: $\hat{A} = \left( W^T V \right)^{-1} W^T AV, \hat{N}_k = \left( W^T V \right)^{-1} W^T N_k V, \hat{B} = \left( W^T V \right)^{-1} W^T B, \hat{C} = CV$
7: end while
8: $A^{opt} = \hat{A}, N_k^{opt} = \hat{N}_k, B^{opt} = \hat{B}, C^{opt} = \hat{C}$
A Heat Transfer Model

- 2-dimensional heat distribution
  \[\Omega = (0, 1) \times (0, 1)\]  
  \[x_t = \Delta x\] in \(\Omega\)  
  \[n \cdot \nabla x = u_{1,2,3}(x - 1)\] on \(\Gamma_1, \Gamma_2, \Gamma_3\)  
  \[x = u_4\] on \(\Gamma_4\)

- boundary control by spraying intensities of a cooling fluid

- spatial discretization \(k \times k\)-grid
  \[\Rightarrow \dot{x} \approx A_1 x + \sum_{i=1}^{3} N_i x u_i + B u\]
  \[\Rightarrow A_2 = 0\]

- output: \[y = \frac{1}{k^2} \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}\]
A Heat Transfer Model

Comparison of relative $\mathcal{H}_2$-error for $n = 10000$

![Graph showing comparison of relative $\mathcal{H}_2$-error for different reduced system dimensions.](image-url)

- Bilinear IRKA
- Balanced Truncation
Outlook

To do:

- investigate possible two-sided reduction methods for quadratic-bilinear systems.

Note that $V \in \mathbb{R}^{n \times q}$ in general is dense $\Rightarrow$ computation of $\hat{A}_2 = V^T A_2 V \otimes V$ might cause problems, $\Rightarrow$ find approximations:

$$A_2 \approx g_1 \otimes G_1 + \ldots + g_r \otimes G_r,$$

with $g_i^T \in \mathbb{R}^n$, $G_i \in \mathbb{R}^{n \times n}$ and $r \ll n$.

Lyapunov-based reduction possible?

Thank you for your attention!
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