Model Reduction for Complex Dynamical Systems
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Nonlinear Model Reduction via Quadratic Bilinear Control Systems

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NETWORK THEORY

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Outline

Motivation

Given a large-scale state-nonlinear control system of the form

$$\boxed{\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + bu(t), \\ y(t) = cx(t), \quad x(0) = x_0, \end{cases}}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear and $b, c^T \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$.

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Common Reduction Techniques

Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model: $[x(t_1), x(t_2), \ldots, x(t_N)] =: X$,
- perform SVD of snapshot matrix: $X = VSW^T \approx V_{\hat{n}} S_{\hat{n}} W_{\hat{n}}^T$.
- Reduction by POD-Galerkin projection: $\dot{\hat{x}} = V_{\hat{n}}^T f(V_{\hat{n}} \hat{x}) + V_{\hat{n}}^T Bu$.
- Requires evaluation of f
  $\rightsquigarrow$ discrete empirical interpolation [Sorensen/Chaturantabut '09].
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Trajectory Piecewise Linear (TPWL)

- Linearize f along trajectory,
- reduce resulting linear systems,
- construct reduced model by weighting sum of linear systems.
- Requires simulation of original model and several linear reduction steps, many heuristics.

State-Space Representation of QBDAEs

We will consider quadratic-bilinear SISO systems of the form

$$E\,\dot{x} = A_1\,x + A_2\,x \otimes x + N\,xu + b\,u$$
$$y = c\,x$$

where $E, A_1, N \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times n^2}$ (Hessian tensor), $b, c^T \in \mathbb{R}^n$.

- A large class of nonlinear control-affine systems can be transformed into the above type of control system.
- The transformation is exact, but a slight increase of the state dimension has to be accepted.
- Input-output behavior can be characterized by generalized transfer functions $\rightsquigarrow$ enables us to use Krylov-based reduction techniques.

Transformation via McCormick Relaxation

Theorem [Gu'09]

Assume that the state equation of a nonlinear system Σ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking derivatives and adding algebraic equations, respectively, Σ can be transformed into a system of quadratic-bilinear DAEs of dimension $N > n$.

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  $\dot{z}_2 = \frac{2 \cdot x_1 \cdot z_1 \cdot z_2}{2 \cdot z_2} = x_1 \cdot z_1.$

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- comparison of terms $\alpha^i, i = 1, 2, \dots$ leads to series of systems

$$\begin{aligned} E\dot{x}_1 &= A_1 x_1 + Bu, \\ E\dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + N x_1 u, \\ E\dot{x}_3 &= A_1 x_3 + A_2 \left(x_1 \otimes x_2 + x_2 \otimes x_1\right) + N x_2 u \\ &\vdots \end{aligned}$$

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- although i-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms x_j, $j < i$, are interpreted as pseudo-inputs.

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$$\begin{aligned} H_3(s_1,s_2,s_3) = \frac{1}{3!}C\left((s_1+s_2+s_3)E - A_1\right)^{-1} \\ \Big[N\big(G_2(s_1,s_2)+G_2(s_2,s_3)+G_2(s_1,s_3)\big) \\ + A_2\big(G_1(s_1)\otimes G_2(s_2,s_3) + G_1(s_2)\otimes G_2(s_1,s_3) \\ + G_1(s_3)\otimes G_2(s_1,s_3) + G_2(s_2,s_3)\otimes G_1(s_1) \\ + G_2(s_1,s_3)\otimes G_1(s_2) + G_2(s_1,s_2)\otimes G_1(s_3)\big)\Big]. \end{aligned}$$

Characterization via Multimoments

For simplicity, focus on the first two transfer functions. For $H_1(s_1)$, choosing σ and making use of the Neumann lemma leads to

$$H_1(s_1) = \sum_{i=0}^{\infty} C \underbrace{\left((A_1 - \sigma E)^{-1} E\right)^i (A_1 - \sigma E)^{-1} B \, (s_1 - \sigma)^i}_{m^i_{s_1,\sigma}}.$$

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Similarly, specifying an expansion point (τ, ξ) yields

$$H_2(s_1, s_2) = \frac{1}{2} \sum_{i=0}^{\infty} C \left((A_1 - (\tau + \xi)E)^{-1} E\right)^i (A_1 - (\tau + \xi)E)^{-1} \, (s_1 + s_2 - \tau - \xi)^i \cdot$$

$$\left[A_2 \left(\sum_{j=0}^{\infty} m^j_{s_1,\tau} \otimes \sum_{k=0}^{\infty} m^k_{s_2,\xi} + \sum_{k=0}^{\infty} m^k_{s_2,\xi} \otimes \sum_{j=0}^{\infty} m^j_{s_1,\tau} \right) + N \left(\sum_{p=0}^{\infty} m^p_{s_1,\tau} + \sum_{p=0}^{\infty} m^q_{s_2,\xi} \right) \right]$$

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for $i = 1 : q$

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&\qquad \textbf{for } j = 1 : \min(q-i+1, i)\\
&\qquad\quad Z_i = \mathcal{K}_{q-i-j+2}\left((A_1 - 2\sigma E)^{-1}E,\right.\\
&\qquad\qquad\qquad \left.(A_1 - 2\sigma E)^{-1}A_2(U_i \otimes U_j + U_j \otimes U_i)\right),
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U_i denoting the i-*th* column of U.

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U_i denoting the i-*th* column of U. Set $V = \mathrm{orth}([U, W, Z])$ and construct $\hat{\Sigma}$ by the Galerkin-Projection $\mathcal{P} = VV^T$:

$$\hat{A}_1 = V^T A_1 V \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{A}_2 = V^T A_2 V \otimes V \in \mathbb{R}^{\hat{n}\times\hat{n}^2},$$

$$\hat{N} = V^T N V \in \mathbb{R}^{\hat{n}\times\hat{n}}, \quad \hat{B} = V^T B \in \mathbb{R}^{\hat{n}}, \quad \hat{C}^T = V^T C \in \mathbb{R}^{\hat{n}}.$$

The FitzHugh-Nagumo System

- FitzHugh-Nagumo system modeling a neuron

[Chaturantabut, Sorensen '09]

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1], \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where
$\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ g = 0.05,\ i_0(t) = 5 \cdot 10^4 t^3 \exp(-15t)$

- original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$
- 3D phase space

Jet Diffusion Flame Model

Next, let us focus on a nonlinear PDE arising in jet diffusion flame models [Galbally, Willcox '09]

$$\frac{\partial w}{\partial t} + U \cdot \nabla w - \nabla(\kappa \nabla w) + f(w) = 0, \quad (x,t) \in (0,1) \times (0,T),$$

with Arrhenius type term $f(w) = Aw(c-w)e^{-\frac{E}{d-w}}$ and constant parameters U, A, E, c, d, κ. Again define initial and boundary conditions:

$$\begin{aligned} w(x,0) &= 0, \quad x \in [0,1], \\ w(0,t) &= u(t), \quad t \geq 0, \\ w(1,t) &= 0, \quad t \geq 0, \\ w_{center} &= \begin{bmatrix} \mathbf{0} & 1 & \mathbf{0} \end{bmatrix}. \end{aligned}$$

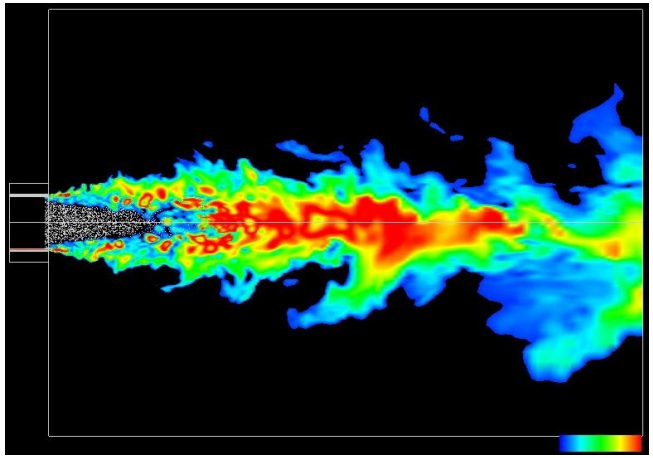
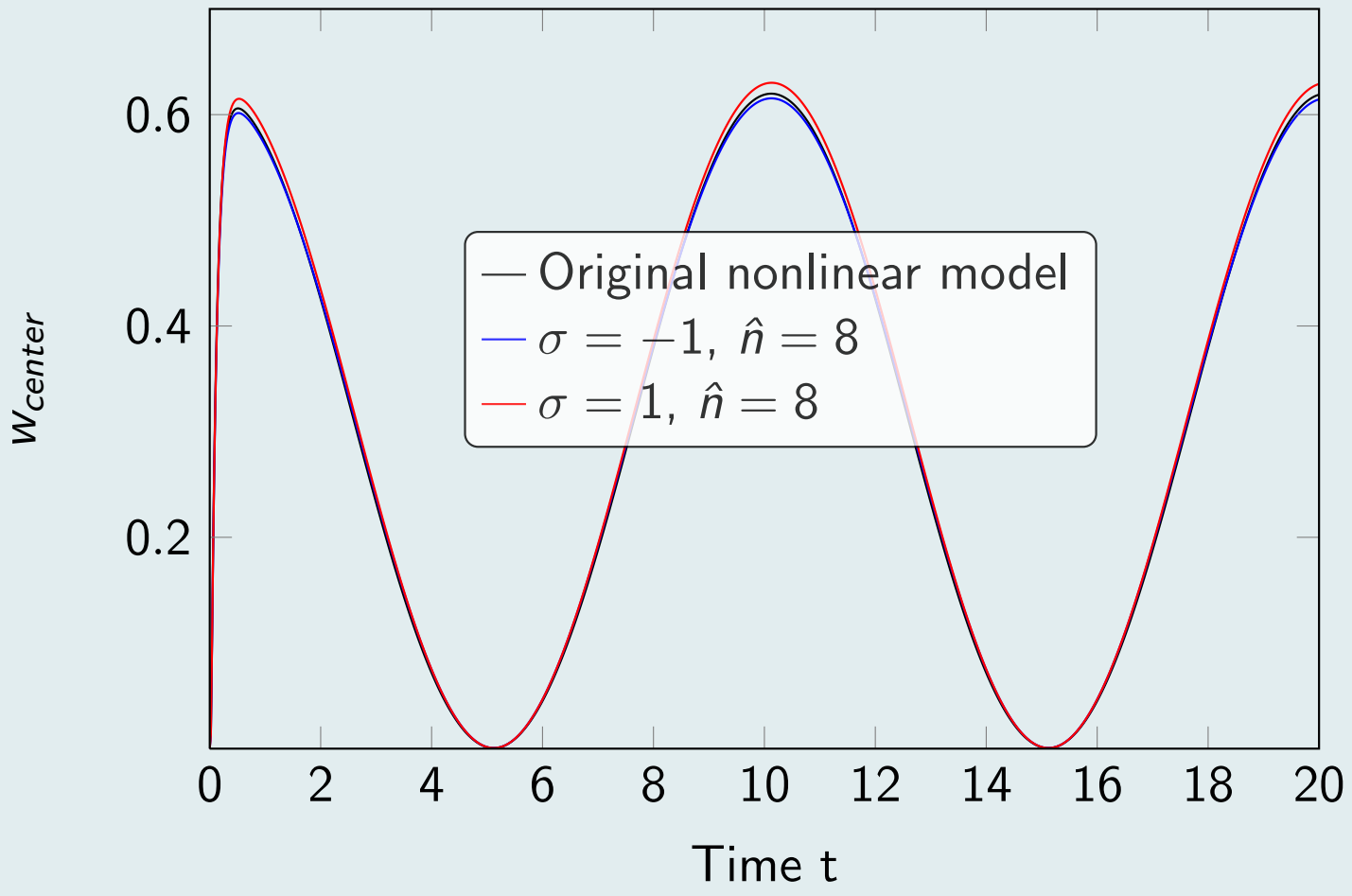


Figure: [Kurose]

Jet Diffusion Flame Model

Transient responses for $k = 1500$ and $u(t) = \frac{1}{2}\cos(\frac{\pi t}{5} + 1)$



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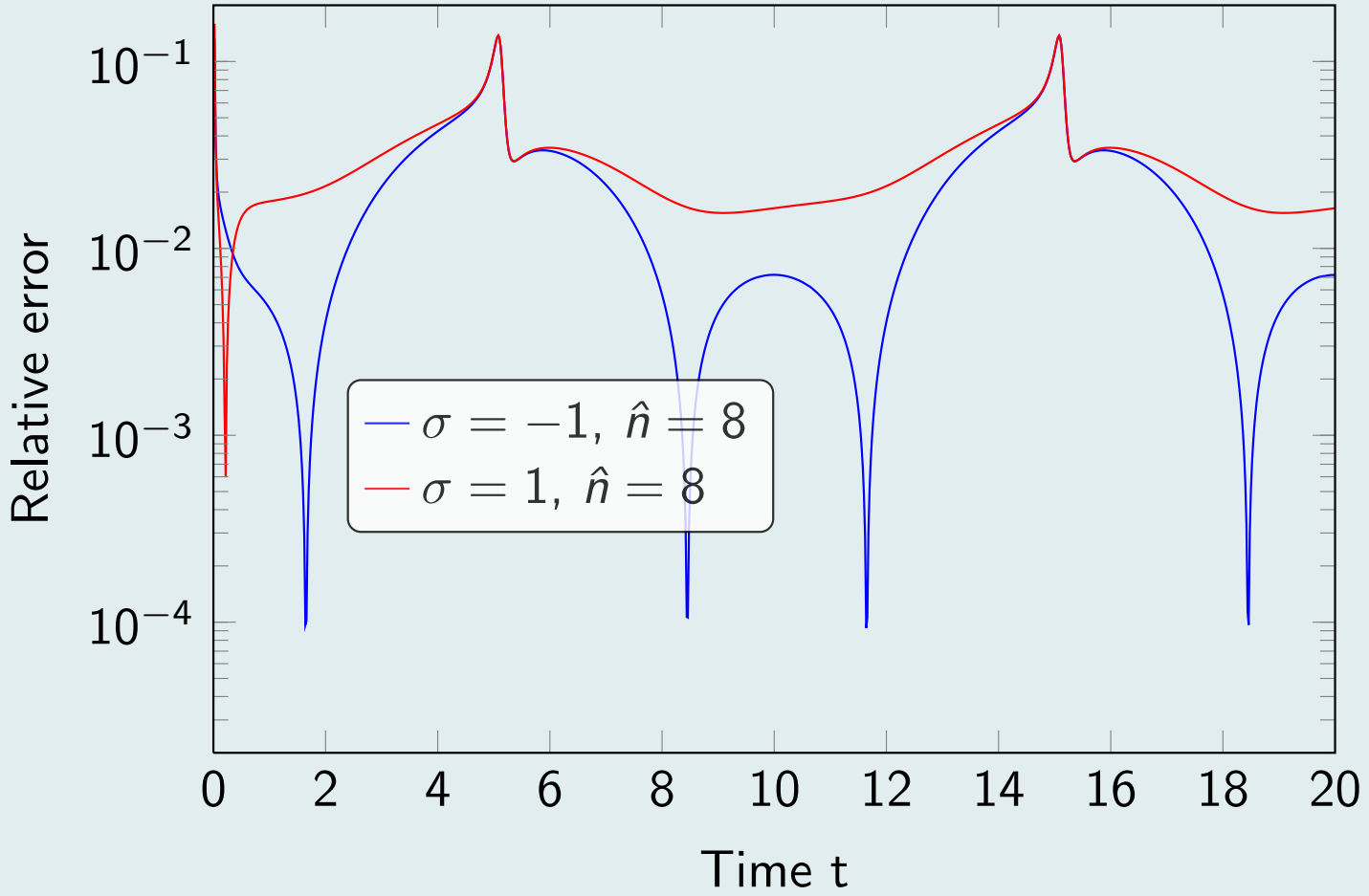
Relative errors for $k = 1500$ and $u(t) = \frac{1}{2}\cos(\frac{\pi t}{5} + 1)$



State-Space and Output Representation

Let us now focus on the special case of bilinear control systems:

$$\Sigma : \quad \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i x(t) u(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases}$$

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Output Characterization (SISO): Volterra series

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{i-1}} h(t_1, \dots, t_k) u(t - t_1 - \dots - t_k) \cdots u(t - t_k) dt_i \cdots dt_1,$$

with kernels $h(t_1, \dots, t_k) = Ce^{At_k} N \cdots e^{At_2} N e^{At_1} B$.

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Let us now focus on the special case of bilinear control systems:

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where $A, N_i \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n}$.

Output Characterization (SISO): Volterra series

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{i-1}} h(t_1, \dots, t_k) u(t - t_1 - \dots - t_k) \cdots u(t - t_k) dt_i \cdots dt_1,$$

with kernels $h(t_1, \dots, t_k) = Ce^{At_k} N \cdots e^{At_2} N e^{At_1} B$.
Multivariable Laplace-transform (SISO):

$$H_i(s_1, \dots, s_i) = C(s_i I - A)^{-1} N \cdots (s_2 I - A)^{-1} N (s_1 I - A)^{-1} B.$$

$\mathcal{H}_2$-Norm for Bilinear Systems

One possible generalization of the known linear $\mathcal{H}_2$-norm is given by:

$$||\Sigma||^2_{\mathcal{H}_2} := \operatorname{tr}\left(\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{(2\pi)^k}\,\overline{H_k(i\omega_1,\ldots,i\omega_k)}H_k^T(i\omega_1,\ldots,i\omega_k)\right),$$

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It can be shown that we can alternatively compute

$$||\Sigma||_{\mathcal{H}_2}^2 = (\operatorname{vec}(I_p))^T\,(C\otimes C)\left(-A\otimes I - I\otimes A - \sum_{k=1}^{m} N_k\otimes N_k\right)^{-1}(B\otimes B)\operatorname{vec}(I_m).$$

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In order to find an $\mathcal{H}_2$-optimal reduced system, we define the error system Σ^{err} as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A}\end{bmatrix},\quad N_k^{err} = \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k\end{bmatrix},\quad B^{err} = \begin{bmatrix} B \\ \hat{B}\end{bmatrix},\quad C^{err} = \begin{bmatrix} C & -\hat{C}\end{bmatrix}.$$

Necessary $\mathcal{H}_2$-Optimality Conditions

Let us assume $\hat{\Sigma}$ is given by its eigenvalue decomposition:

$$\hat{A} = R\Lambda R^{-1}, \quad \tilde{N}_k = R^{-1}\hat{N}_k R, \quad \tilde{B} = R^{-1}\hat{B}, \quad \tilde{C} = \hat{C}R.$$

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&(\mathrm{vec}(I_p))^T \left(e_i e_j^T \otimes C\right) \begin{pmatrix} -\lambda_1 I - A & & \\ & \ddots & \\ & & -\lambda_{\hat{n}} I - A \end{pmatrix}^{-1} \begin{pmatrix} B\tilde{B}_1^T \\ \vdots \\ B\tilde{B}_{\hat{n}}^T \end{pmatrix} \\
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$$H(-\lambda_j)\tilde{B}_j^T = \hat{H}(-\lambda_j)\tilde{B}_j^T$$

A Bilinear IRKA Approach

Algorithm 1 Bilinear IRKA

Input: A, N_k, B, C, $\hat{A}$, $\hat{N}_k$, $\hat{B}$, $\hat{C}$
Output: A^{opt}, N_k^{opt}, B^{opt}, C^{opt}

1: **while** (change in $\Lambda > \epsilon$) **do**
2: $R\Lambda R^{-1} = \hat{A}$, $\tilde{B} = R^{-1}\hat{B}$, $\tilde{C} = \hat{C}R$, $\tilde{N}_k = R^{-1}\hat{N}_k R$
3: $\operatorname{vec}(V) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k \otimes N_k\right)^{-1} \left(\tilde{B} \otimes B\right) \operatorname{vec}(I_m)$
4: $\operatorname{vec}(W) = \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k^T\right)^{-1} \left(\tilde{C}^T \otimes C^T\right) \operatorname{vec}(I_p)$
5: $V = \operatorname{orth}(V)$, $W = \operatorname{orth}(W)$
6: $\hat{A} = \left(W^T V\right)^{-1} W^T A V$, $\hat{N}_k = \left(W^T V\right)^{-1} W^T N_k V$,
$\hat{B} = \left(W^T V\right)^{-1} W^T B$, $\hat{C} = CV$
7: **end while**
8: $A^{opt} = \hat{A}$, $N_k^{opt} = \hat{N}_k$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

A Heat Transfer Model

- 2-dimensional heat distribution [Benner, Saak '05]
- boundary control by spraying intensities of a cooling fluid

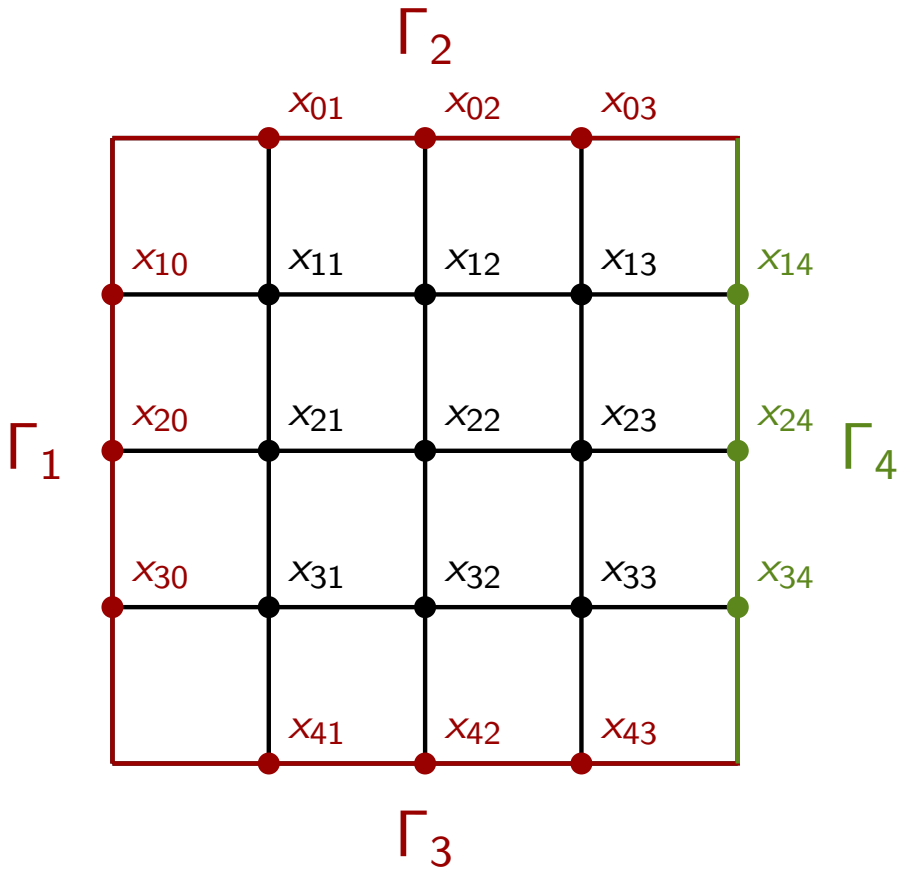
$$\begin{aligned} \Omega &= (0,1) \times (0,1) & \\ x_t &= \Delta x & \text{in } \Omega \\ n \cdot \nabla x &= u_{1,2,3}(x-1) & \text{on } \Gamma_1, \Gamma_2, \Gamma_3 \\ x &= u_4 & \text{on } \Gamma_4 \end{aligned}$$

- spatial discretization $k \times k$-grid

$$\Rightarrow \dot{x} \approx A_1 x + \sum_{i=1}^{3} N_i x u_i + Bu$$

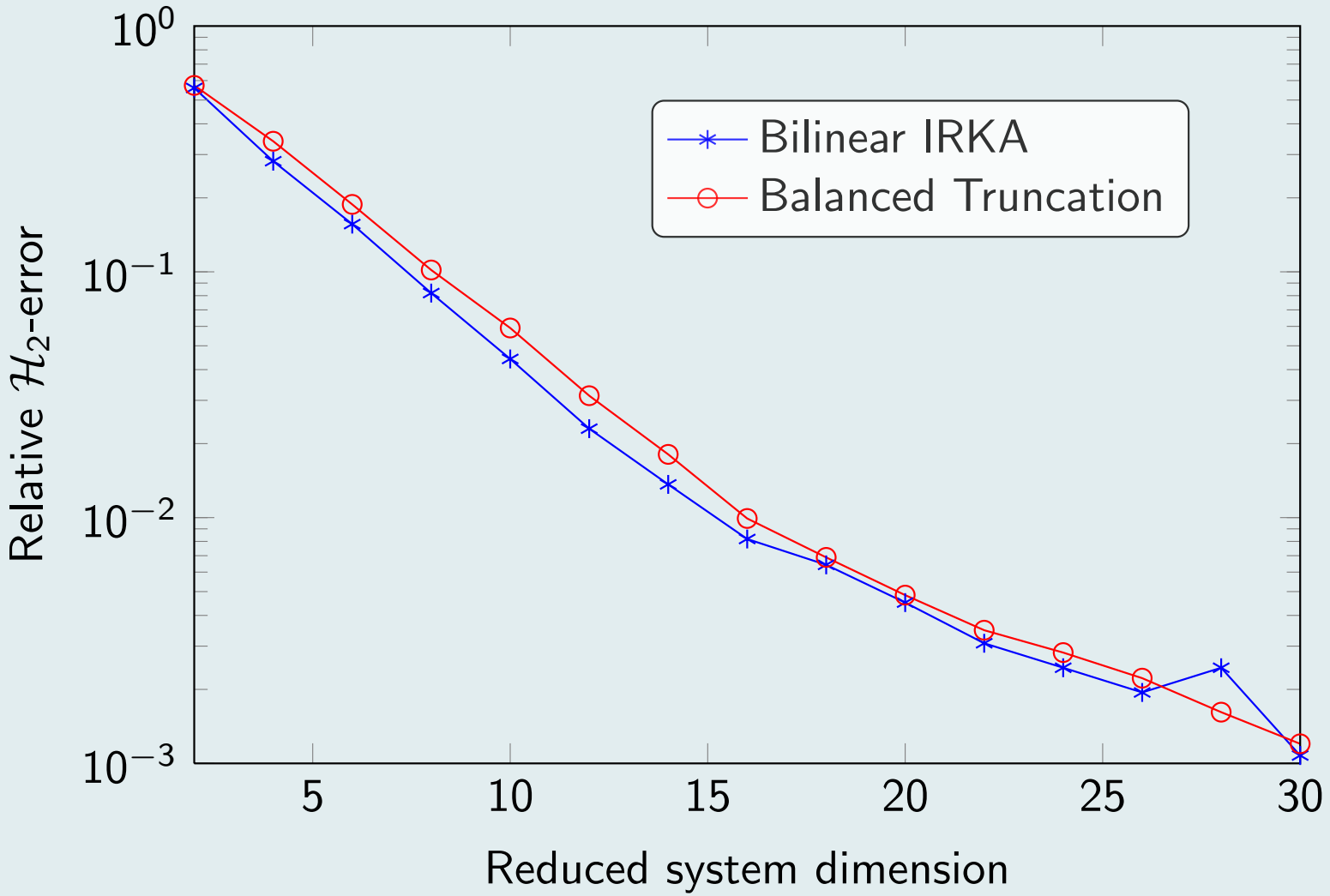
$$\Rightarrow A_2 = 0$$

- output: $y = \frac{1}{k^2}\begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$



A Heat Transfer Model

Comparison of relative $\mathcal{H}_2$-error for $n = 10000$



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Thank you for your attention!