On parametrized systems

Cosmin Ionita, Sanda Lefteriu and Thanos Antoulas

Rice University and Jacobs University

email: aca@rice.edu
URL: www.ece.rice.edu/˜aca

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- Balanced truncation offers something in addition, namely, a **trade-off between accuracy and complexity**.
- In the following, we will present a method in the same spirit.

Outline

The Loewner matrix: 1D case

- $\mathcal{P}_n$: space of all polynomials of degree at most n. $\Rightarrow \dim(\mathcal{P}_n) := n+1$.
- *Monomial basis*: s^i, $i = 0, 1, \cdots, n$.
- Given $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n+1$: $\lambda_i \neq \lambda_j$, $i \neq j$,

$$\mathbf{q}_i(s) := \Pi_{i' \neq i}(s - \lambda_{i'}),\ i = 1, \cdots, n+1,$$

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- It readily follows that

$$\mathbf{g}(\lambda_i) = \mathbf{w}_i,$$

since $\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s-\lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s-\lambda_i}} = \frac{\sum_i \beta_i \mathbf{q}_i(s)}{\sum_i \alpha_i \mathbf{q}_i(s)}$, where $\beta_i = \alpha_i \mathbf{w}_i$.

The free parameters α_i, can be specified so that

$$\mathbf{g}(\mu_j) = \mathbf{v}_j,\ j = 1, \cdots, r,$$

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where $(\mu_j, \mathbf{v}_j)$, $\mu_i \neq \mu_j$, are given. Thus

$$\mathbb{L}\mathbf{c} = 0,\ \ \mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{n+1}}{\mu_1 - \lambda_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_r - \mathbf{w}_1}{\mu_r - \lambda_1} & \cdots & \frac{\mathbf{v}_r - \mathbf{w}_{n+1}}{\mu_r - \lambda_{n+1}} \end{bmatrix} \in \mathbb{C}^{r\times(n+1)},\ \mathbf{c} = \begin{bmatrix} \alpha_1, \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}.$$

$\mathbb{L}$: **Loewner matrix** with

row array $(\mu_j, \mathbf{v}_j)$, $j = 1, \cdots, r$, and

column array $(\lambda_i, \mathbf{w}_i)$, $i = 1, \cdots, n+1$.

Properties of the Loewner matrix

There is a bijective correspondence between rational functions and Loewner matrices. Recall: *complexity* or *(McMillan) degree* is the maximum between the degrees of the numerator and denominator. Let

$$P = \{(x_i, y_i) : \; x_i, y_i \in \mathbb{C}, \; i = 1, \cdots, N\}.$$

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This array is partitioned in a *column array* P_c and in a *row array* P_r, where

$$P_c = \{(\lambda_i, \mathbf{w}_i) : \; i = 1, \cdots, k\}, \;\; P_r = \{(\mu_i, \mathbf{v}_i) : \; i = 1, \cdots, p\}.$$

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It is assumed that $P = P_c \cup P_r$. To this partitioning we associate a $p \times k$ Loewner matrix

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}, \ \ i = 1, \cdots, p, \ \ j = 1, \cdots, k.$$

Main property

Given $\mathbf{g}$ and the array of points P, where $y_i = \mathbf{g}(x_i)$, let $\mathbb{L}$ be a $p \times k$ Loewner matrix for some partitioning P_c, P_r of P. Then

$$\mathbf{p}, \mathbf{k} \geq \deg \mathbf{g} \quad \Rightarrow \quad \operatorname{rank} \mathbb{L} = \deg \mathbf{g}.$$

Consequently, every square Loewner matrix of size $\deg \mathbf{g}$, is non-singular.

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Conversely: we define the *rank* of an array P:

$$\operatorname{rank} P := \max_{\mathbb{L}} [rank \mathbb{L}] =: q,$$

where the maximum is taken over all possible Loewner matrices which can be built from P. It follows that the rank of all Loewner matrices which have at least q rows and columns is equal to q. Assuming that $2q < N$, let $\mathbf{c} = [c_1, \cdots, c_{q+1}]^*$, be such that $\mathbb{L}\mathbf{c} = 0$, for any $\mathbb{L}$ of size $q \times (q+1)$. In this case we can attach to $\mathbb{L}$ a rational function $\mathbf{g}$ by means of the formula $\sum_{i=1}^{q+1} c_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0$. The main result is that if all possible square Loewner matrices of size q are non-singular, $\mathbf{g}$ is the unique interpolant of degree q.

Approximate interpolation and model reduction in 1D

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The above considerations show that in the Loewner matrix framework

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A recently obtained error expression shows that the 2-norm of the interpolation error is proportional to the first neglected singular value of $\mathbb{L}$.

2D Lagrange bases and Loewner matrix

• $\mathcal{P}_{n,m}$: space of all polynomials in two indeterminates, say s and t, so that degree with respect to s is at most n and degree with respect to t is at most m $\Rightarrow \dim \mathcal{P}_{n,m} = (n+1)(m+1)$.

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- *monomial basis*: $s^i t^j$, $i = 0, 1, \cdots, n$, $j = 0, 1, \cdots, m$.
- For a *Lagrange basis*, we need distinct (for simplicity) complex numbers λ_i, $i = 1, \cdots, n+1$, π_j, $j = 1, \cdots, m+1$:

$$\mathbf{q}_{i,j}(s,t) := \prod_{i' \neq i}(s - \lambda_{i'}) \prod_{j' \neq j}(t - \pi_{j'}), \quad i = 1, \cdots, n+1, \quad j = 1, \cdots, m+1.$$

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$$\mathbf{q}_{i,j}(s,t) := \prod_{i' \neq i} (s - \lambda_{i'}) \prod_{j' \neq j} (t - \pi_{j'}), \quad i = 1, \cdots, n+1, \quad j = 1, \cdots, m+1.$$

- Consider $\mathbf{g}(s,t)$ defined by

$$\sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \alpha_{i,j} \frac{\mathbf{g} - \mathbf{w}_{i,j}}{(s - \lambda_i)(t - \pi_j)} = 0, \quad \alpha_{i,j} \neq 0.$$

Since we can write

$$\mathbf{g}(s,t) = \frac{\sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \alpha_{i,j} \mathbf{w}_{i,j}\, \mathbf{q}_{i,j}(s,t)}{\sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \alpha_{i,j}\, \mathbf{q}_{i,j}(s,t)}.$$

it follows that **g** satisfies the interpolation conditions $\mathbf{g}(\lambda_i, \pi_j) = \mathbf{w}_{i,j}$.

- As in 1D, the parameters $\alpha_{i,j}$ can be determined so that **g** satisfies additional interpolation conditions:

$$\mathbf{g}(\mu_i, \nu_j) = \mathbf{v}_{i,j},\ i = 1, \cdots, p+1,\ j = 1, \cdots, r+1,$$

where $(\mu_j, \nu_j; \mathbf{v}_{i,j})$, are given triples of complex numbers; it is assumed for simplicity that the μ_j are distinct and and not equal to any λ_i; similarly all ν_j are distinct and not equal to any π_i.

- Consider the two arrays

$$P_c := \{(\lambda_j, \pi_i; \mathbf{w}_{j,i}) : \ i = 1, \cdots, n', \ j = 1, \cdots, m'\},$$
$$P_r := \{(\mu_l, \nu_k; \mathbf{v}_{l,k}) : \ k = 1, \cdots, p', \ l = 1, \cdots, r'\},$$

for some positive integers n', m', p', r', and let

$$\ell_{i,j}^{k,l} := \frac{\mathbf{v}_{k,l} - \mathbf{w}_{i,j}}{(\mu_k - \lambda_i)(\nu_l - \pi_j)}.$$

The associated Loewner matrix has entries $\ell_{i,j}^{k,l}$, where the superscripts k, l determine the rows in the ordering $(1,1), \ (1,2), \cdots, (2,1), \ (2,2), \cdots$, and the the subscripts i, j determine the columns in the same ordering. The *two-variable Loewner matrix* $\mathbb{L}$ defined above has dimension $p'r' \times n'm'$.

Example. Consider $\phi(s,t) = \frac{1}{s\,t-1}$. Goal: reconstruct this rational function from measurements. We choose:

$$[\lambda_1, \lambda_2, \mu_3, \mu_4] = \left[\ 2,\quad \tfrac{1}{2},\quad \tfrac{3}{2},\quad 3\ \right],$$
$$[\pi_1, \pi_2, \nu_3, \nu_4] = \left[\ -\tfrac{1}{2},\quad -\tfrac{3}{2},\quad -1,\quad -2\ \right].$$

The corresponding values are:

$$\mathbf{w}_{11} = -\tfrac{1}{2},\quad \mathbf{w}_{12} = -\tfrac{1}{4},\quad \mathbf{w}_{21} = -\tfrac{4}{5},\quad \mathbf{w}_{22} = -\tfrac{4}{7},$$
$$\mathbf{v}_{11} = -\tfrac{2}{5},\quad \mathbf{v}_{12} = -\tfrac{1}{4},\quad \mathbf{v}_{21} = -\tfrac{2}{11},\quad \mathbf{v}_{22} = -\tfrac{1}{7}.$$

The Loewner matrix with

- column indices (λ_1, π_1), (λ_1, π_2), (λ_2, π_1), (λ_2, π_2), and
- row indices (μ_3, ν_3), (μ_3, ν_4), (μ_4, ν_3), (μ_4, ν_4), is:

$$\mathbb{L} = \left[\begin{array}{cc|cc}
\frac{\mathbf{v}_{11}-\mathbf{w}_{11}}{(\mu_1-\lambda_1)(\nu_1-\pi_1)} & \frac{\mathbf{v}_{11}-\mathbf{w}_{12}}{(\mu_1-\lambda_1)(\nu_1-\pi_2)} & \frac{\mathbf{v}_{11}-\mathbf{w}_{21}}{(\mu_1-\lambda_2)(\nu_1-\pi_1)} & \frac{\mathbf{v}_{11}-\mathbf{w}_{22}}{(\mu_1-\lambda_2)(\nu_1-\pi_2)} \\
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\end{array}\right] =$$

$$\left[\begin{array}{cc|cc} \frac{2}{5} & \frac{3}{5} & -\frac{4}{5} & \frac{12}{35} \\ \frac{1}{3} & 0 & -\frac{11}{30} & -\frac{9}{14} \\ \hline -\frac{1}{2} & 0 & -\frac{11}{25} & \frac{9}{35} \\ -\frac{5}{21} & -\frac{3}{14} & -\frac{92}{525} & -\frac{12}{35} \end{array}\right].$$

It has rank equal to 3 and $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbf{c} = [8,\ -16,\ -5,\ 7]^*.$$

Thus:

$$8\,\frac{\mathbf{g}+\frac{1}{2}}{(s-2)\left(t+\frac{1}{2}\right)} - 16\,\frac{\mathbf{g}+\frac{1}{4}}{(s-2)\left(t+\frac{3}{2}\right)} - 5\,\frac{\mathbf{g}+\frac{4}{5}}{\left(s-\frac{1}{2}\right)\left(t+\frac{1}{2}\right)} + 7\,\frac{\mathbf{g}+\frac{4}{7}}{\left(s-\frac{1}{2}\right)\left(t+\frac{3}{2}\right)} = 0.$$

which implies $\mathbf{g}(s,t) = \phi(s,t) = \frac{1}{s\,t-1}$ (original function recovered).

Structure of 2D Loewner matrices

Assuming that $n' = n + 1$ and $m' = m + 1$, the entries of **c** will be rearranged, and the quantities $\mathbf{c}^{(i)}$, $\mathbf{c}_{(j)}$ defined:

$$\mathbb{A} := \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n+1} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m+1,1} & \alpha_{m+1,2} & \cdots & \alpha_{m+1,n+1} \end{bmatrix}, \quad \mathbf{c}^{(i)} := \mathbb{A}(i,:), \; \mathbf{c}_{(j)} := \mathbb{A}(:,j).$$

Thus $\mathbf{c} = \mathrm{vec}\,\mathbb{A}$, where 'vec' denotes the *vectorization* of a matrix obtained by stacking its rows into a column vector, i.e. $\mathbf{c}^* = [\mathbf{c}^{(1)} \; \cdots \; \mathbf{c}^{(m+1)}]$.

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In the sequel we will denote by $\mathbb{L}_{\pi_i}$ and $\mathbb{L}_{\lambda_j}$, single-variable Loewner matrices of appropriate dimensions obtained by sampling the 1D rational functions $\mathbf{g}(s, \pi_i)$ and $\mathbf{g}(\lambda_j, t)$, respectively.

Main Lemma

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(a1) If $\mathbb{L}\mathbf{c} = 0$, its components $\mathbf{c}_{(j)}$, and $\mathbf{c}^{(i)}$, satisfy:

$$\mathbb{L}_{\pi_j}\mathbf{c}_{(j)} = 0,\ j = 1, \cdots, m' \ \text{ and } \ \mathbf{c}^{(i)}\mathbb{L}_{\lambda_i} = 0,\ i = 1, \cdots, n'.$$

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Remark: The computational complexity can be reduced to

$$n' \cdot (m')^3 + (n')^3 \ \text{ or } \ m' \cdot (n')^3 + (m')^3,$$

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(b) Given $\mathbf{g}(s, t)$ of complexity (n, m), let the arrays above be defined by means of its samples. Assume also that $n' \geq n + 1$ and $m' \geq m + 1$. The rank of the associated Loewner matrix $\mathbb{L}$ is:

$$\mathrm{rank}\,\mathbb{L} \ = \ \mathbf{n'm' - (n' - n)(m' - m)}$$

Remarks.

(a). In 1D: $\operatorname{rank} \mathbb{L} = n' - (n' - n) = n$, that is, it is independent of the number of measurements (interpolation data). In 2D the rank of $\mathbb{L}$ still encodes the information about the complexity n, m, of interpolants. This property makes $\mathbb{L}$ the fundamental tool for Data-driven PMOR.

(b) The tableau below pictorially displays the samples needed to define the various Loewner matrices.

$s \backslash t$	π_1	$\cdots$	$\pi_{m'}$	ν_1	$\cdots$	$\nu_{m'}$
λ_1	$\mathbf{w}_{1,1}$	$\cdots$	$\mathbf{w}_{1,m'}$	$\mathbf{w}_{1,m'+1}$	$\cdots$	$\mathbf{w}_{1,2m'}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\lambda_{n'}$	$\mathbf{w}_{n',1}$	$\cdots$	$\mathbf{w}_{n',m'}$	$\mathbf{w}_{n',m'+1}$	$\cdots$	$\mathbf{w}_{n',m'+1}$
μ_1	$\mathbf{w}_{n'+1,1}$	$\cdots$	$\mathbf{w}_{n'+1,m'}$	$\mathbf{v}_{1,1}$	$\cdots$	$\mathbf{v}_{1,m'}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mu_{n'}$	$\mathbf{w}_{2n',1}$	$\cdots$	$\mathbf{w}_{2n',m'}$	$\mathbf{v}_{n',1}$	$\cdots$	$\mathbf{v}_{n',m'}$

Example. Consider the 2D rational function

$$\mathbf{g}(s,t) = \frac{s^2}{s-t+1}.$$

We wish to recover $\mathbf{g}$ from samples at the following points:

$$(s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6,\ s_7,\ s_8) = \left(\ 0,\ \ \tfrac{1}{2},\ \ 1,\ \ \tfrac{3}{2},\ \ 2,\ \ \tfrac{3}{4},\ \ 3,\ \ \tfrac{5}{2},\ \right),$$
$$(t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8) = \left(\ -\tfrac{1}{4},\ \ -\tfrac{1}{2},\ \ -\tfrac{3}{4},\ \ -1,\ \ -\tfrac{5}{4},\ \ -\tfrac{3}{2},\ \ 0,\ \ -2\ \right).$$

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Hence $n=2$, $m=1$, $n'=m'=4$. The Loewner matrix $\mathbb{L}_{16}$, with column array formed from the first four s_i and t_j, has dimension 16×16. According to the main lemma its rank is $n'm'-(n'-n)(m'-m)=10$, while its null space has dimension $(n'-n)(m'-m)=6$ and is spanned by the following vectors:

$$
\left(\begin{array}{c|ccc|ccc}
(\mathbf{s_1}, \mathbf{t_1}) & -\frac{1}{2} & -\frac{5}{11} & -\frac{5}{12} & -\frac{5}{6} & -\frac{10}{13} & -\frac{5}{7} \\
(\mathbf{s_1}, \mathbf{t_2}) & \frac{3}{5} & 0 & 0 & 1 & 0 & 0 \\
(\mathbf{s_1}, \mathbf{t_3}) & 0 & \frac{7}{11} & 0 & 0 & \frac{14}{13} & 0 \\
(\mathbf{s_1}, \mathbf{t_4}) & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{8}{7} \\
\hline
(\mathbf{s_2}, \mathbf{t_1}) & \frac{7}{5} & \frac{14}{11} & \frac{7}{6} & \frac{7}{4} & \frac{21}{13} & \frac{3}{2} \\
(\mathbf{s_2}, \mathbf{t_2}) & -\frac{8}{5} & 0 & 0 & -2 & 0 & 0 \\
(\mathbf{s_2}, \mathbf{t_3}) & 0 & -\frac{18}{11} & 0 & 0 & -\frac{27}{13} & 0 \\
(\mathbf{s_2}, \mathbf{t_4}) & 0 & 0 & -\frac{5}{3} & 0 & 0 & -\frac{15}{7} \\
\hline
(\mathbf{s_3}, \mathbf{t_1}) & -\frac{9}{10} & -\frac{9}{11} & -\frac{3}{4} & 0 & 0 & 0 \\
(\mathbf{s_3}, \mathbf{t_2}) & 1 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{s_3}, \mathbf{t_3}) & 0 & 1 & 0 & 0 & 0 & 0 \\
(\mathbf{s_3}, \mathbf{t_4}) & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
(\mathbf{s_4}, \mathbf{t_1}) & 0 & 0 & 0 & -\frac{11}{12} & -\frac{11}{13} & -\frac{11}{14} \\
(\mathbf{s_4}, \mathbf{t_2}) & 0 & 0 & 0 & 1 & 0 & 0 \\
(\mathbf{s_4}, \mathbf{t_3}) & 0 & 0 & 0 & 0 & 1 & 0 \\
(\mathbf{s_4}, \mathbf{t_4}) & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{c|ccc|ccc}
(\mathbf{s_1},\mathbf{t_1}) & -\frac{1}{2} & -\frac{5}{11} & -\frac{5}{12} & -\frac{5}{6} & -\frac{10}{13} & -\frac{5}{7} \\
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(\mathbf{s_1},\mathbf{t_4}) & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{8}{7} \\
\hline
(\mathbf{s_2},\mathbf{t_1}) & \frac{7}{5} & \frac{14}{11} & \frac{7}{6} & \frac{7}{4} & \frac{21}{13} & \frac{3}{2} \\
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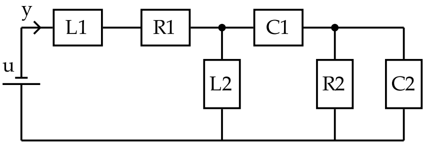
(\mathbf{s_2},\mathbf{t_3}) & 0 & -\frac{18}{11} & 0 & 0 & -\frac{27}{13} & 0 \\
(\mathbf{s_2},\mathbf{t_4}) & 0 & 0 & -\frac{5}{3} & 0 & 0 & -\frac{15}{7} \\
\hline
(\mathbf{s_3},\mathbf{t_1}) & -\frac{9}{10} & -\frac{9}{11} & -\frac{3}{4} & 0 & 0 & 0 \\
(\mathbf{s_3},\mathbf{t_2}) & 1 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{s_3},\mathbf{t_3}) & 0 & 1 & 0 & 0 & 0 & 0 \\
(\mathbf{s_3},\mathbf{t_4}) & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
(\mathbf{s_4},\mathbf{t_1}) & 0 & 0 & 0 & -\frac{11}{12} & -\frac{11}{13} & -\frac{11}{14} \\
(\mathbf{s_4},\mathbf{t_2}) & 0 & 0 & 0 & 1 & 0 & 0 \\
(\mathbf{s_4},\mathbf{t_3}) & 0 & 0 & 0 & 0 & 1 & 0 \\
(\mathbf{s_4},\mathbf{t_4}) & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Thus the 2D Lagrange bases involved are obtained from all combinations of the 1D bases formed by (s_1, s_2, s_3), (s_1, s_2, s_4), and (t_1, t_2), (t_1, t_3), (t_1, t_4) $\Rightarrow$ 6 Lagrange bases associated with $\mathbb{L}_{16}$. All 6 resulting rational functions are equal to $\mathbf{g}$. ■

Circuit example



Given the circuit above, all elements have unit value except L_2 = parameter t. Using the voltages across the capacitors and the currents through the inductors as state variables, we obtain the equations: $\mathbf{E\dot{x}} = \mathbf{Ax} + \mathbf{Bu}$, $\mathbf{y} = \mathbf{Cx}$, where

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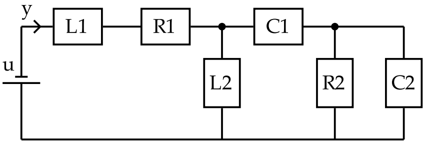

Given the circuit above, all elements have unit value except L_2 = parameter t. Using the voltages across the capacitors and the currents through the inductors as state variables, we obtain the equations: $\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

while $\mathbf{C} = \mathbf{B}^*$. For comparison later on, we notice that (the inverse of the resolvent) $\mathbf{\Phi}(s,t) = s\mathbf{E} - \mathbf{A}$, contains the product of the two variables st.

Thus the transfer function depends on two variables, namely the (complex) frequency s, and the value t of one of the inductors:

$$\mathbf{g}(s,t) = \mathbf{C}\,[s\mathbf{E} - \mathbf{A}]^{-1}\,\mathbf{B} = \frac{t\,s^3 + t\,s^2 + 2\,s + 1}{s^4\,t + 2\,s^3\,t + 3\,s^2\,t + 2\,s^2 + s\,t + 3\,s + 1}.$$

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• We will reconstruct this system using measurements at the following frequencies and values of the parameter:

$$[s_1,\;\; s_2,\;\; s_3,\;\; s_4,\;\; s_5,\;\; t_1,\;\; t_2] = \left[\; 0,\;\; \tfrac{1}{2},\;\; 1,\;\; \tfrac{3}{2},\;\; 2,\;\; 0,\;\; -\tfrac{1}{2}\;\right]$$

$$[s_6,\;\; s_7,\;\; s_8,\;\; s_9,\;\; s_{10},\;\; t_3,\;\; t_4] = \left[\; -\tfrac{1}{4},\;\; -\tfrac{1}{2},\;\; -1,\;\; -\tfrac{3}{2},\;\; -2,\;\; 1,\;\; \tfrac{1}{2}\;\right]$$

The resulting 10×10 Loewner matrix is:

$\mathbb{L}$ and **c** are:

$$\begin{bmatrix} -\frac{268}{73} & -\frac{536}{219} & -\frac{1096}{657} & -\frac{5072}{5183} & -\frac{414}{365} & -\frac{1088}{1825} & -\frac{2216}{2555} & -\frac{23312}{1533} & -\frac{1388}{1971} & -\frac{349}{657} \\ -\frac{792}{169} & -\frac{396}{169} & -\frac{3728}{1521} & -\frac{36904}{35997} & -\frac{1468}{845} & -\frac{2656}{4225} & -\frac{8016}{5915} & -\frac{26760}{1183} & -\frac{5080}{4563} & -\frac{1975}{3042} \\ -2 & -\frac{4}{3} & -\frac{4}{3} & -\frac{56}{71} & -1 & -\frac{8}{15} & -\frac{4}{5} & -\frac{40}{3} & -\frac{2}{3} & -\frac{1}{2} \\ -4 & -2 & -\frac{8}{3} & -\frac{84}{71} & -2 & -\frac{4}{5} & -\frac{8}{5} & -20 & -\frac{4}{3} & -\frac{3}{4} \\ 2 & \frac{4}{3} & \frac{10}{9} & \frac{172}{213} & \frac{3}{4} & \frac{3}{5} & \frac{14}{25} & -\frac{148}{15} & \frac{4}{9} & \frac{1}{4} \\ 6 & 3 & \frac{32}{9} & \frac{400}{213} & \frac{5}{2} & \frac{7}{5} & \frac{48}{25} & -\frac{72}{5} & \frac{14}{9} & \frac{17}{24} \\ \frac{82}{73} & \frac{164}{219} & \frac{148}{219} & \frac{7784}{15549} & \frac{173}{365} & \frac{2168}{5475} & \frac{132}{365} & -\frac{1816}{219} & \frac{446}{1533} & \frac{473}{3066} \\ \frac{228}{89} & \frac{114}{89} & \frac{424}{267} & \frac{5492}{6319} & \frac{506}{445} & \frac{1532}{2225} & \frac{392}{445} & -\frac{1100}{89} & \frac{1340}{1869} & \frac{745}{2492} \\ \frac{10}{13} & \frac{20}{39} & \frac{94}{195} & \frac{1668}{4615} & \frac{9}{26} & \frac{58}{195} & \frac{122}{455} & -\frac{1948}{273} & \frac{17}{78} & \frac{23}{208} \end{bmatrix}, \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ -\frac{3}{2} \\ \frac{71}{64} \\ \frac{9}{2} \\ -\frac{15}{8} \\ -5 \\ -\frac{1}{64} \\ \frac{15}{8} \\ 1 \end{bmatrix}$$

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$$
\begin{bmatrix}
-\frac{268}{73} & -\frac{536}{219} & -\frac{1096}{657} & -\frac{5072}{5183} & -\frac{414}{365} & -\frac{1088}{1825} & -\frac{2216}{2555} & -\frac{23312}{1533} & -\frac{1388}{1971} & -\frac{349}{657} \\
-\frac{792}{169} & -\frac{396}{169} & -\frac{3728}{1521} & -\frac{36904}{35997} & -\frac{1468}{845} & -\frac{2656}{4225} & -\frac{8016}{5915} & -\frac{26760}{1183} & -\frac{5080}{4563} & -\frac{1975}{3042} \\
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\end{bmatrix},
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\end{bmatrix}
$$

$\mathbb{L}\mathbf{c} = 0$ and $\operatorname{rank}\mathbb{L} = 9$. To recover **g** solve the following equation for ϕ:

$$
\frac{\phi-1}{8ts} - \frac{3\phi-2}{2t(s-\frac{1}{2})} + \frac{9\phi-\frac{9}{2}}{2t(s-1)} - \frac{5\phi-2}{t(s-\frac{3}{2})} + \frac{15\phi-5}{8t(s-2)}
- \frac{\phi-1}{8s(t+\frac{1}{2})} + \frac{71\phi-58}{64(s-\frac{1}{2})(t+\frac{1}{2})} - \frac{15\phi-12}{8(s-1)(t+\frac{1}{2})} - \frac{\phi+38}{64(s-\frac{3}{2})(t+\frac{1}{2})} + \frac{\phi-\frac{1}{8}}{(s-2)(t+\frac{1}{2})} = 0.
$$

Indeed $\phi(s,t) = \mathbf{g}(s,t)$.

Approximate interpolation and model reduction in 2D

Conclusion

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the singular values of $\mathbb{L}$

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MOR procedure

- From $\mathrm{rank}\,\mathbb{L} = n'm' - (n'-n)(m'-m)$ determine the desired complexities n and m.
- Build a model either in numerator/denominator format, or by means of a description realization (next section).

Outline

One variable case: $\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\beta_i}{s-\lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s-\lambda_i}}$

We define:

$$\mathbf{J}_{\text{lag}}(s;\lambda_i,n)=\begin{bmatrix} s-\lambda_1 & \lambda_2-s & & & \\ s-\lambda_1 & 0 & \lambda_3-s & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots \\ s-\lambda_1 & & & 0 & \lambda_{n+1}-s \end{bmatrix} \in\mathbb{R}^{n\times(n+1)}[s], \tag{1}$$

and $\mathbf{a}=[\alpha_0,\alpha_1,\cdots,\alpha_n]=\mathbf{c}^T$ $(\mathbb{L}\mathbf{c}=\mathbf{0})$, $\mathbf{b}=[\beta_0,\beta_1,\cdots,\beta_n]$ $(\beta_i=\alpha_i\mathbf{w}_i)$.

Lemma

$$\widehat{\mathbf{C}}=\mathbf{b},\quad \widehat{\mathbf{\Phi}}(s)=\begin{pmatrix} \mathbf{J}_{\text{lag}}(s;\lambda_i,n) \\ \mathbf{a} \end{pmatrix},\quad \widehat{\mathbf{B}}=\mathbf{e}_{n+1}, \tag{2}$$

is an R-controllable and R-observable order $n+1$ realization of $\mathbf{g}$.

Corollary

$$\mathbf{C}=[\,\mathbf{0}\mid -1],\quad \mathbf{\Phi}(s)=\left[\begin{array}{c|c} \widehat{\mathbf{\Phi}}(s) & \mathbf{0} \\ \hline \widehat{\mathbf{C}} & 1 \end{array}\right],\quad \mathbf{B}=\left[\begin{array}{c} \widehat{\mathbf{B}} \\ \hline 0 \end{array}\right], \tag{3}$$

is an R-controllable and R-observable realization of size $n+2$.

Two variable case: $\mathbf{g}(s,p)=\frac{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\frac{\beta_{i,j}}{(s-\lambda_i)(p-\pi_j)}}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\frac{\alpha_{i,j}}{(s-\lambda_i)(p-\pi_j)}}$

$$\mathbb{A}:=\begin{bmatrix}\alpha_{00} & \alpha_{10} & \cdots & \alpha_{n0}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_{0m} & \alpha_{1m} & \cdots & \alpha_{nm}\end{bmatrix}, \mathbb{B}:=\begin{bmatrix}\beta_{00} & \beta_{10} & \cdots & \beta_{n0}\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{0m} & \beta_{1m} & \cdots & \beta_{nm}\end{bmatrix}.$$

Theorem

$$\boldsymbol{\Phi}(s,p)=\left[\begin{array}{c|c|c}\mathbf{J}_{\text{lag}}(s;\lambda_i,n) & \mathbf{0} & \mathbf{0}\\ \hline \mathbb{A} & \mathbf{J}^*_{\text{lag}}(p;\pi_j,m) & \mathbf{0}\\ \hline \mathbb{B} & \mathbf{0} & [\mathbf{J}^*_{\text{lag}}(p;\pi_j,m),\ \mathbf{p}]\end{array}\right], \mathbf{B}=\left[\begin{array}{c}\mathbf{0}\\ \hline \mathbf{p}\\ \hline \mathbf{0}\end{array}\right],$$

$$\mathbf{C}=[\,\mathbf{0}\mid\mathbf{0}\mid -\mathbf{e}_{m+1}^T],$$

is a realization of $\mathbf{g}(s,p)$ *of dimension* $n+2m+2$*, R-controllable and R-observable where* $\mathbf{p}\in\mathbb{C}^{m+1}$ *is a vector such that* $[\mathbf{J}^*_{\text{lag}}(p;\pi_j,m),\ \mathbf{p}]$ *is a unimodular matrix in p. The entries of this vector can be chosen as:* $p_i=1/\Pi_{j=1,j\neq i}^{m+1}(\pi_i-\pi_j)$*, in which case* $\det[\mathbf{J}^*_{\text{lag}}(p;\pi_j,m),\ \mathbf{p}]=(-1)^m$*.*

Example: $\mathbf{H}(s,p) = \frac{1+2s+3p+4sp}{5+6s+7p+8sp}$

- Lagrange basis: $(s+1)(p+1)$, $(s+1)(p+2)$, $(s+2)(p+1)$, $(s+2)(p+2)$
- $\mathbb{A} = \begin{bmatrix} 0 & -2 \\ -1 & 11 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 & -2 \\ -1 & 7 \end{bmatrix}$
- $\mathbf{g}(s,p) =$

$$\left[\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & -1 \end{array}\right] \left[\begin{array}{cc|c|cc} s+1 & -s-2 & 0 & 0 & 0 \\ \hline 0 & -2 & p+1 & 0 & 0 \\ -1 & 11 & -p-2 & 0 & 0 \\ \hline 0 & -2 & 0 & p+1 & -1 \\ -1 & 7 & 0 & -p-2 & 1 \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \hline -1 \\ 1 \\ \hline 0 \\ 0 \end{array}\right]$$

Numerical examples - Circuit (continued)

Measurements at the following frequencies and values of the parameter:

$$[s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ p_1,\ p_2] = \left[0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ 0,\ -\frac{1}{2}\right]$$

Nullspace of $\mathbb{L}$: $\mathbf{c} = \left[\ \frac{1}{8}\quad -\frac{1}{8}\quad -\frac{3}{2}\quad \frac{71}{64}\quad \frac{9}{2}\quad -\frac{15}{8}\quad -5\quad -\frac{1}{64}\quad \frac{15}{8}\quad 1\ \right].$

$$\mathbf{\Phi}(s,p) = \left[\begin{array}{ccccc|c|cc} s & \frac{1}{2}-s & 0 & 0 & 0 & 0 & 0 & 0 \\ s & 0 & 1-s & 0 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & \frac{3}{2}-s & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 2-s & 0 & 0 & 0 \\ \hline \frac{1}{8} & -\frac{3}{2} & \frac{9}{2} & -5 & \frac{15}{8} & p & 0 & 0 \\ -\frac{1}{8} & \frac{71}{64} & -\frac{15}{8} & -\frac{1}{64} & 1 & -p-\frac{1}{2} & 0 & 0 \\ \hline \frac{1}{8} & -1 & \frac{9}{4} & -2 & \frac{5}{8} & 0 & p & -1 \\ -\frac{1}{8} & \frac{29}{32} & -\frac{3}{2} & \frac{19}{32} & \frac{1}{8} & 0 & -p-\frac{1}{2} & 1 \end{array}\right], \mathbf{B} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ -1 \\ \hline 0 \\ 0 \end{array}\right], \mathbf{C}^T = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ -1 \end{array}\right]$$

Numerical examples - Circuit

By adding row 5 to 6, the submatrix of $\mathbf{\Phi}$ consisting of rows & columns $6, 7, 8$ is unimodular $\Rightarrow$ Schur complement $\Rightarrow$ realization of dimension 5:

$$\mathbf{\Phi}_5(s,p)=\left[\begin{array}{ccccc} s & \frac{1}{2}-s & 0 & 0 & 0 \\ s & 0 & 1-s & 0 & 0 \\ s & 0 & 0 & \frac{3}{2}-s & 0 \\ s & 0 & 0 & 0 & 2-s \\ \hline \frac{1}{8} & -\frac{25}{32}p-\frac{3}{2} & \frac{21}{4}p+\frac{9}{2} & -\frac{321}{32}p-5 & \frac{23}{4}p+\frac{15}{8} \end{array}\right], \mathbf{B}_5=\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \end{array}\right],$$

while

$$\mathbf{C}_5(p)=\left[\begin{array}{ccccc} \frac{1}{8} & -\frac{3}{16}p-1 & \frac{3}{2}p+\frac{9}{4} & -\frac{45}{16}p-2 & \frac{3}{2}p+\frac{5}{8} \end{array}\right].$$

Numerical examples - Convection Diffusion

- Consider the 2-dimensional convection-diffusion equation in $\Omega = (0,1)^2$

$$\frac{\partial \mathbf{x}}{\partial t}(t,\xi) = \Delta \mathbf{x}(t,\xi) + p \cdot \nabla \mathbf{x}(t,\xi) + \mathbf{b}(\xi)\mathbf{u}(t), \xi \in \Omega,$$

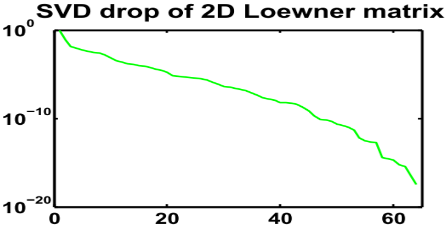
$$\mathbf{x}(t,\xi) = \mathbf{0}, \xi \in \partial\Omega$$

- A finite-difference discretization leads to an order $n = 900$ parametric system (note that $m = 900$):

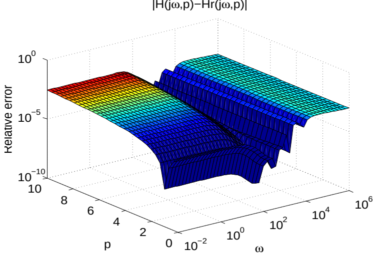
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + p\mathbf{A}_1\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),\ \mathbf{x}(0) = 0$$
$$\mathbf{H}(s,p) = \mathbf{C}(s\mathbf{I} - \mathbf{A} - p\mathbf{A}_1)^{-1}\mathbf{B}$$

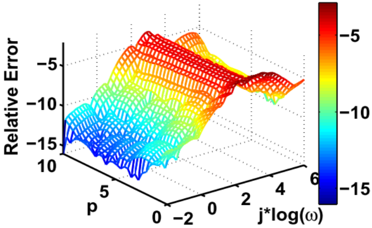
- We reduce to an order $n = m = 7$ system by selecting measurements in s logarithmically spaced between $j10^{-2}$ and $j10^{6}$ and measurements in p linearly spaced between 0 and 10.

Numerical examples - Convection Diffusion



Error evaluated for 50 freq vals logarithmically spaced $10^{-2} - 10^{6}$, and 30 parameter vals linearly spaced $0 - 10$:
$\Rightarrow$ max error 1.3e-3.





Outline

1D Loewner-MOR

- sample a 1D rational function $\mathbf{H}(s)$ of degree n:

$$\mathbf{w}_j := \mathbf{H}(\lambda_j), \;\; \mathbf{v}_i := \mathbf{H}(\mu_i), \;\; \lambda_j \neq \mu_i.$$

- construct the 1D Loewner matrix, $\mathbb{L} \in \mathbb{C}^{N \times n'}$, $N > n'$,

$$\mathbb{L} := \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{n'}}{\mu_1 - \lambda_{n'}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_{n'} - \mathbf{w}_1}{\mu_{n'} - \lambda_1} & \cdots & \frac{\mathbf{v}_{n'} - \mathbf{w}_{n'}}{\mu_{n'} - \lambda_{n'}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_N - \mathbf{w}_1}{\mu_N - \lambda_1} & \cdots & \frac{\mathbf{v}_N - \mathbf{w}_{n'}}{\mu_N - \lambda_{n'}} \end{bmatrix}.$$

- then, for $n' > n$, the degree can be read off from the rank of $\mathbb{L}$ $\mathrm{rank}\mathbb{L} = n$.
- from $\mathbf{x} \in ker\mathbb{L}$ original system recovered as $\mathbf{H}(s) = \frac{\sum_{j=1}^{n'} \frac{\mathbf{w}_j \mathbf{x}_j}{s - \lambda_j}}{\sum_{j=1}^{n'} \frac{\mathbf{x}_j}{s - \lambda_j}}$.

1D Loewner-MOR

- **model order reduction:**

  construct a **'tall'** Loewner matrix $\mathbb{L} \in \mathbb{C}^{N\times(k+1)}, \quad N > k,$

$$\mathbb{L} := \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{k+1}}{\mu_1 - \lambda_{k+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_{k+1} - \mathbf{w}_1}{\mu_{k+1} - \lambda_1} & \cdots & \frac{\mathbf{v}_{k+1} - \mathbf{w}_{k+1}}{\mu_{k+1} - \lambda_{k+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_N - \mathbf{w}_1}{\mu_N - \lambda_1} & \cdots & \frac{\mathbf{v}_N - \mathbf{w}_{k+1}}{\mu_N - \lambda_{k+1}} \end{bmatrix}$$

  with number of columns $k < n$.

- then $\mathbb{L}$ has full column rank

$$\text{rank}\mathbb{L} = k + 1,$$

  and

$$\mathbb{L}\mathbf{x} = \mathbf{b} \neq \mathbf{0}, \quad \forall\, \mathbf{x} \in \mathbb{C}^{k+1}.$$

1D Loewner-MOR

- for each row of $\mathbb{L}\mathbf{x} = \mathbf{b}$, rewrite $\mathbf{v}_i$ as: $\mathbf{v}_i = \frac{\sum_{j=1}^{k+1} \frac{\mathbf{w}_j \mathbf{x}_j}{\mu_i - \lambda_j}}{\sum_{j=1}^{k+1} \frac{\mathbf{x}_j}{\mu_i - \lambda_j}} + \frac{\mathbf{b}_i}{\sum_{j=1}^{k+1} \frac{\mathbf{x}_j}{\mu_i - \lambda_j}}.$
- define **reduced model** of order k as: $\mathbf{G}(s) := \frac{\sum_{j=1}^{k+1} \frac{\mathbf{w}_j \mathbf{x}_j}{s - \lambda_j}}{\sum_{j=1}^{k+1} \frac{\mathbf{x}_j}{s - \lambda_j}}.$

  the **error** is: $\mathbf{H}(\lambda_j) - \mathbf{G}(\lambda_j) = 0, \quad j = 1, \ldots, k+1,$

$$\mathbf{H}(\mu_i) - \mathbf{G}(\mu_i) = \frac{\mathbf{b}_i}{\sum_{j=1}^{k+1} \frac{\mathbf{x}_j}{\mu_i - \lambda_j}} := \mathbf{d}_i, \quad i = 1, \ldots, N.$$

- for $\mathbb{L} \in \mathbb{C}^{N \times (k+1)}$, find $\mathbf{b} \in \mathbb{C}^N$ and $\mathbf{x} \in \mathbb{C}^{k+1}$ such that

$$\min_{\mathbb{L}\mathbf{x} = \mathbf{b}} \|\mathbf{d}\|_2.$$

1D Loewner-MOR

- eliminate **b** and re-write the minimization problem as

$$\min_{\mathbf{x}} \ \|\widehat{\mathbf{d}}\|_2, \quad \text{with} \ \ \widehat{\mathbf{d}}_i := \sum_{j=1}^{k+1} \frac{(\mathbf{v}_i - \mathbf{w}_j)\mathbf{x}_j}{\mu_i - \lambda_j} \Big/ \sum_{j=1}^{k+1} \frac{\mathbf{x}_j}{\mu_i - \lambda_j}$$

- solve a sub-optimal minimization problem

$$\min_{\mathbf{x}} \ \|\widetilde{\mathbf{d}}\|_2, \qquad \text{with} \ \ \widetilde{\mathbf{d}}_i := \sum_{j=1}^{k+1} \frac{(\mathbf{v}_i - \mathbf{w}_j)\mathbf{x}_j}{\mu_i - \lambda_j} \ \Leftrightarrow \ \min_{\mathbf{x}} \ \|\mathbb{L}\mathbf{x}\|_2.$$

- compute SVD of Loewner matrix ($\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}^*$).
- **solution:**

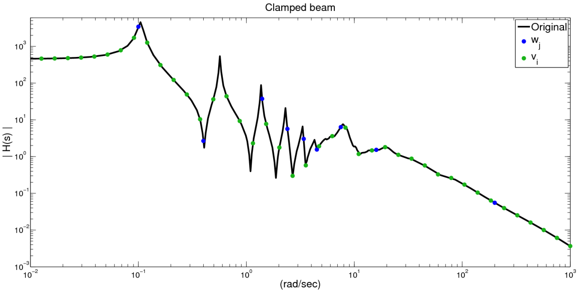
$$\mathbf{x} = \text{right-singular vector corresponding to } \sigma_{k+1}.$$

- **error formula:**

$$\mathbf{H}(\mu_i) - \mathbf{G}(\mu_i) = \sigma_{k+1} \frac{\mathbf{y}_i}{\sum_{j=1}^{k+1} \frac{\mathbf{x}_j}{\mu_i - \lambda_j}},$$

where $\mathbf{y}$ = left-singular vector corresponding to σ_{k+1}.

1D Loewner-MOR

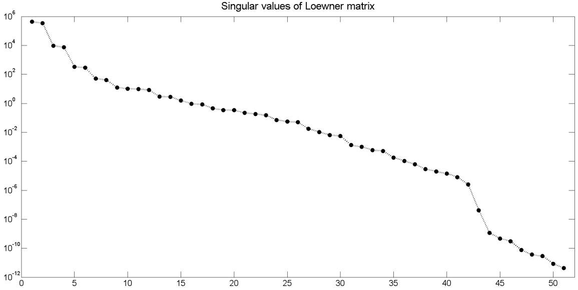


- original system of order $n = 348$, take measurements

  $\mathbf{w}_j := \mathbf{H}(\lambda_j),\ \lambda_j \in \{0, \pm.1j, \pm.4j, \pm1.4j, \pm2.4j, \pm3.3j, \pm4.5j, \pm7.5j, \pm16j, \pm250j\},$
  $\mathbf{v}_i := \mathbf{H}(\mu_i),\ \mu_i \in \{\pm 10^{-3}j, \ldots, \pm 10^{3}j\}.$

- construct 'tall' 1D Loewner matrix $\mathbb{L} \in \mathbb{C}^{100\times18}$, $\quad \mathbb{L}_{i,j} := \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$.

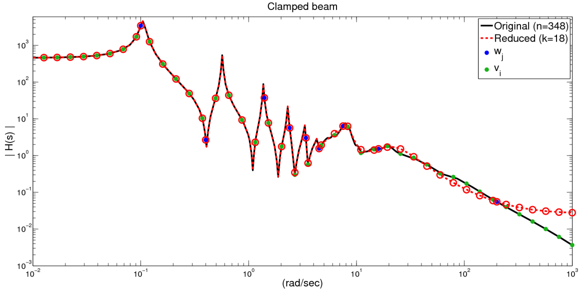
1D Loewner-MOR



- $(\mathbf{y}, \mathbf{x})$ = (left-,right-) singular vectors for smallest singular value of $\mathbb{L}$, σ_{18}.
- reduced model of order $k = 18$

$$\mathbf{G}(s) := \frac{\sum_{j=1}^{k+1} \frac{\mathbf{w}_j \mathbf{x}_j}{s-\lambda_j}}{\sum_{j=1}^{k+1} \frac{\mathbf{x}_j}{s-\lambda_j}}.$$

1D Loewner-MOR



- by construction, measurements $\mathbf{w}_j$ interpolated exactly.
- for measurements $\mathbf{v}_i$, maximum point-wise error

$$\max_i \left|\mathbf{H}(\mu_i) - \mathbf{G}(\mu_i)\right| = \sigma_{k+1} \cdot \max_i \left| \frac{\mathbf{y}(i)}{\sum_j \frac{\mathbf{x}_j}{\mu_i - \lambda_j}} \right| = 0.5.$$

2D Loewner-MOR

- sample a 2D rational function $\mathbf{H}(s,t)$ of degree (n,m):

$$\mathbf{w}_{i,j} := \mathbf{H}(\lambda_i,\pi_j),\ \ \mathbf{v}_{k,\ell} := \mathbf{H}(\mu_k,\nu_\ell),\ \ \lambda_i \neq \mu_k,\ \pi_j \neq \nu_\ell.$$

- construct 2D Loewner matrix, $\mathbb{L} \in \mathbb{C}^{N\times(n'm')}$, $(NM) \geq (n'm')$

$$\mathbb{L} := \begin{bmatrix} \frac{\mathbf{v}_{1,1}-\mathbf{w}_{1,1}}{(\mu_1-\lambda_1)(\nu_1-\pi_1)} & \cdots & \frac{\mathbf{v}_{1,1}-\mathbf{w}_{n',m'}}{(\mu_1-\lambda_{n'})(\nu_1-\pi_{m'})} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_{n',m'}-\mathbf{w}_{1,1}}{(\mu_{n'}-\lambda_1)(\nu_{m'}-\pi_1)} & \cdots & \frac{\mathbf{v}_{n',m'}-\mathbf{w}_{n',m'}}{(\mu_{n'}-\lambda_{n'})(\nu_{m'}-\pi_{m'})} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_{N,M}-\mathbf{w}_{1,1}}{(\mu_N-\lambda_1)(\nu_M-\pi_1)} & \cdots & \frac{\mathbf{v}_{N,M}-\mathbf{w}_{n',m'}}{(\mu_N-\lambda_{n'})(\nu_M-\pi_{m'})} \end{bmatrix}.$$

- then for $n' > n, m' > m$, $\ \ \mathit{dim\ ker}\ \mathbb{L} = (n'-n)(m'-m)$.

- and from $\mathbf{x} \in \mathit{ker}\mathbb{L}$, recover $\mathbf{H}(s,t) = \dfrac{\sum_{i=1}^{n'}\sum_{j=1}^{m'} \frac{\mathbf{w}_{i,j}\mathbf{x}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}{\sum_{i=1}^{n'}\sum_{j=1}^{m'} \frac{\mathbf{x}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}.$

2D Loewner-MOR

- **model order reduction:**

  construct a **'tall'** 2D Loewner matrix $\mathbb{L} \in \mathbb{C}^{(NM)\times(p+1)(q+1)}$

$$\mathbb{L} := \begin{bmatrix} \frac{\mathbf{v}_{1,1}-\mathbf{w}_{1,1}}{(\mu_1-\lambda_1)(\nu_1-\pi_1)} & \cdots & \frac{\mathbf{v}_{1,1}-\mathbf{w}_{p+1,q+1}}{(\mu_1-\lambda_{p+1})(\nu_1-\pi_{q+1})} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_{p+1,q+1}-\mathbf{w}_{1,1}}{(\mu_{p+1}-\lambda_1)(\nu_{q+1}-\pi_1)} & \cdots & \frac{\mathbf{v}_{p+1,q+1}-\mathbf{w}_{p+1,q+1}}{(\mu_{p+1}-\lambda_{p+1})(\nu_{q+1}-\pi_{q+1})} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_{N,M}-\mathbf{w}_{1,1}}{(\mu_N-\lambda_1)(\nu_M-\pi_1)} & \cdots & \frac{\mathbf{v}_{N,M}-\mathbf{w}_{p+1,q+1}}{(\mu_N-\lambda_{p+1})(\nu_M-\pi_{q+1})} \end{bmatrix}.$$

  with $p < n,\ q < m$.

- then $\mathbb{L}$ has full column rank

$$\text{rank}\mathbb{L} = (p+1)(q+1),$$

  and

$$\mathbb{L}\mathbf{x} = \mathbf{b} \neq \mathbf{0}, \quad \forall\, \mathbf{x} \in \mathbb{C}^{(p+1)(q+1)}.$$

2D Loewner-MOR

- define **reduced model** of order (p, q) as:

$$\mathbf{G}(s,t) := \frac{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}\frac{\mathbf{w}_{i,j}\mathbf{x}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}\frac{\mathbf{x}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}.$$

the **error** is:

$$\mathbf{H}(\lambda_i,\pi_j) - \mathbf{G}(\lambda_i,\pi_j) = 0,$$
$$\mathbf{H}(\mu_i,\nu_j) - \mathbf{G}(\mu_i,\nu_j) = \frac{\mathbf{b}_{k,\ell}}{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}\frac{\mathbf{x}_{i,j}}{(\mu_k-\lambda_i)(\nu_\ell-\pi_j)}} := \mathbf{d}_{k,\ell}.$$

- for $\mathbb{L} \in \mathbb{C}^{(NM)\times(p+1)(q+1)}$, find $\mathbf{b} \in \mathbb{C}^{(NM)}$ and $\mathbf{x} \in \mathbb{C}^{(p+1)(q+1)}$ such that

$$\min_{\mathbb{L}\mathbf{x}=\mathbf{b}} \|\mathbf{d}\|_2.$$

2D Loewner-MOR

- eliminate **b** and re-write the minimization problem as $\min_{\mathbf{x}} \ \|\widehat{\mathbf{d}}\|_2$,

$$\widehat{\mathbf{d}}_{k,\ell} := \sum_{i=1}^{p+1}\sum_{j=1}^{q+1} \frac{(\mathbf{v}_{k,\ell} - \mathbf{w}_{i,j})\mathbf{x}_{k,\ell}}{(\mu_k - \lambda_i)(\nu_\ell - \pi_j)} \Bigg/ \sum_{i=1}^{p+1}\sum_{j=1}^{q+1} \frac{\mathbf{x}_{k,\ell}}{(\mu_k - \lambda_i)(\nu_\ell - \pi_j)} .$$

- solve a sub-optimal minimization problem

$$\min_{\mathbf{x}} \ \|\widetilde{\mathbf{d}}\|_2, \ \ \widetilde{\mathbf{d}}_{k,\ell} := \sum_{i=1}^{p+1}\sum_{j=1}^{q+1} \frac{(\mathbf{v}_{k,\ell} - \mathbf{w}_{i,j})\mathbf{x}_{k,\ell}}{(\mu_k - \lambda_i)(\nu_\ell - \pi_j)} \ \Leftrightarrow \ \min_{\mathbf{x}} \ \|\mathbb{L}\mathbf{x}\|_2.$$

- **solution:**

$$\mathbf{x} = \text{right-singular vector corresponding to } \sigma_{(p+1)(q+1)}.$$

- **error formula:**

$$\mathbf{H}(\mu_i, \nu_j) - \mathbf{G}(\mu_i, \nu_j) = \sigma_{(p+1)(q+1)} \frac{\mathbf{y}_{k,\ell}}{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1} \frac{\mathbf{x}_{i,j}}{(\mu_k - \lambda_i)(\nu_\ell - \pi_j)}},$$

where $\mathbf{y}$ = left-singular vector corresponding to $\sigma_{(p+1)(q+1)}$.

2D Loewner-MOR

Example of 2D MOR in the Loewner framework:

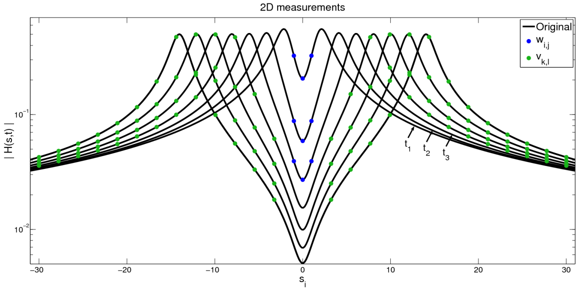
Two-variable rational function of order $(n, m) = (6, 6)$

$$\mathbf{H}(s,t) = \frac{(s^4 + t^4 + 0.5)(s+1)}{(s^4 + t^4)(s^2 + 2s + t^2 + 1)}.$$

- **Given:** measurements $\mathbf{H}(s_i, t_j)$.
- **Goal:** construct reduced model $\mathbf{G}(s,t)$ of order $(p, q) = (2, 2)$ s.t.

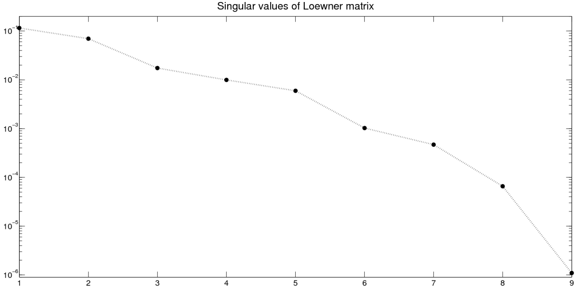
$$\mathbf{G}(s_i, t_j) \approx \mathbf{H}(s_i, t_j).$$

2D Loewner-MOR



- data: $$\begin{aligned}\mathbf{w}_{i,j} &:= \mathbf{H}(\lambda_i, \pi_j), \quad \lambda_i \in \{0, \pm j\},\ \pi_j \in \{2,4,6\},\\ \mathbf{v}_{k,\ell} &:= \mathbf{H}(\mu_k, \nu_\ell), \quad \mu_k \in \{\pm 3.2j, \ldots, \pm 30j\},\ \nu_\ell \in \{8, \ldots, 14\}.\end{aligned}$$
- 2D Loewner matrix $\mathbb{L} \in \mathbb{C}^{104\times 9}$, with entries $[\mathbb{L}] := \left[\frac{\mathbf{v}_{k,\ell} - \mathbf{w}_{i,j}}{(\mu_k - \lambda_i)(\nu_\ell - \pi_j)}\right]$.

2D Loewner-MOR

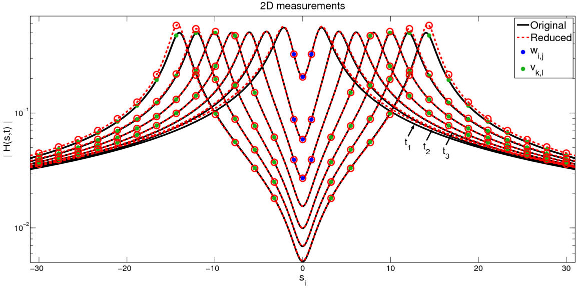


- reduced model of order $(p, q) = (2, 2)$

$$\mathbf{G}(s,t) := \frac{\sum_{i=1}^{p+1} \sum_{j=1}^{q+1} \frac{\mathbf{w}_{i,j}\mathbf{x}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}{\sum_{i=1}^{p+1} \sum_{j=1}^{q+1} \frac{\mathbf{x}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}.$$

- $[x_{1,1},\ x_{1,2},\ \ldots,\ x_{3,3}]^T :=$ right singular vector of $\mathbb{L}$ corresponding to σ_9.

2D Loewner-MOR



- by construction, measurements $\mathbf{w}_{i,j}$ interpolated exactly.
- for measurements $\mathbf{v}_{k,\ell}$, maximum point-wise error

$$\max_{k,\ell} \left| \mathbf{H}(\mu_k,\nu_\ell) - \mathbf{G}(\mu_k,\nu_\ell) \right| = \sigma_9 \cdot \max_{k,\ell} \left| \frac{\mathbf{y}_{k,\ell}}{\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\mathbf{x}_{i,j}}{(\mu_k-\lambda_i)(\nu_\ell-\pi_j)}} \right| = 0.1.$$

Outline

Conclusions

- We presented a data-based model reduction method.
- Based of the Loewner matrix framework.
- Provides trade-off between accuracy and complexity.
- Aposteriori error computation.
- Extension to parametrized systems.

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