Self-Adjoint Differential-Algebraic Equations in Linear-Quadratic Optimal Control Problems

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1. The linear-quadratic optimal control problem

2. The self-conjugate differential-algebraic operator

3. Condensed forms for self-adjoint pairs of matrix functions

4. Self-adjoint DAEs and symplectic flow
1. The linear-quadratic optimal control problem

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The Linear-Quadratic Optimal Control Problem

Minimize a quadratic cost functional

\[ J(x, u) = \frac{1}{2} x^T M_e x + \frac{1}{2} \int_t^\bar{t} \left( x^T W x + x^T S u + u^T S^T x + u^T R u \right) dt \]

subject to the constraint

\[ E \dot{x} = Ax + Bu + f, \quad x(t) = x \in \mathbb{R}^n, \quad (1) \]

with \( I = [t, \bar{t}], \ E, \ A, \ W \in C(I, \mathbb{R}^{n,n}), \ B, \ S \in C(I, \mathbb{R}^{n,m}), \ M_e \in \mathbb{R}^{n,n}, \ f \in C(I, \mathbb{R}^n), \ R \in C(I, \mathbb{R}^{m,m}), \) with \( R = R^T, \ W = W^T, \ M_e = M_e^T \) and continuous functions \( x : I \rightarrow \mathbb{R}^n, \ u : I \rightarrow \mathbb{R}^m. \)

We assume that the DAE (1) is strangeness-free (d-index 1) as a behavior system and that the coefficients are sufficiently smooth.
Necessary optimality conditions are given by the boundary value problem (BVP) (see Kunkel & Mehrmann 2008)

\[
\begin{bmatrix}
0 & E & 0 \\
-E^T & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
d \lambda \\
dx/dt
\end{bmatrix}
= \begin{bmatrix}
0 & A & B \\
A^T + d/dt E^T & W & S \\
B^T & S^T & R
\end{bmatrix}
\begin{bmatrix}
\lambda \\
x
\end{bmatrix}
+ \begin{bmatrix}
f \\
0
\end{bmatrix}
\]

with consistent boundary conditions

\[
x(t) = x, \quad E(t)^T \lambda(t) = M_e x(t).
\]

The associated pair \((\mathcal{E}, \mathcal{A})\) of matrix functions satisfies

\[
\mathcal{E}^T = -\mathcal{E}, \quad \mathcal{A}^T = \mathcal{A} + \frac{d}{dt} \mathcal{E}.
\]

We call such a pair a \textit{self-adjoint} pair of matrix functions.
1. The linear-quadratic optimal control problem

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We can write the optimal control problem as

\[
\frac{1}{2} \mathcal{Q}(z, z) = \min \text{ such that } \mathcal{L}(z) = c, \quad z = \begin{bmatrix} x \\ u \end{bmatrix}, \quad c = \begin{bmatrix} f \\ E(t)^+ E(t)x \end{bmatrix}
\]

\(\triangleright\) where \(\mathcal{Q} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}\) is a symmetric quadratic form defined by

\[
\mathcal{Q}(z, \tilde{z}) = z(\bar{t})^T \begin{bmatrix} M_e & 0 \\ 0 & 0 \end{bmatrix} \tilde{z}(\bar{t}) + \int_I z^T \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \tilde{z} \, dt,
\]

\(\triangleright\) \(\mathcal{L} : \mathbb{Z} \to \mathbb{Y}\) is a linear submersion defined by

\[
\mathcal{L}(z) = \left( E \frac{d}{dt} (E^+ Ex) - (A + E \frac{d}{dt} (E^+ E))x - Bu, \quad E(t)^+ E(t)x(t) \right)
\]

\(\triangleright\) with the Banach spaces \(\mathbb{Z} = \mathbb{X} \times \mathbb{U}\) and

\[
\mathbb{X} = \mathcal{C}^{1}_{E^+ E}(I, \mathbb{R}^n) := \{ x \in \mathcal{C}(I, \mathbb{R}^n), \quad E^+ Ex \in \mathcal{C}^1(I, \mathbb{R}^n) \},
\]

\[
\mathbb{U} = \mathcal{C}(I, \mathbb{R}^m), \quad \mathbb{Y} = \mathcal{C}(I, \mathbb{R}^n) \times \text{range} \ E(t)^T.
\]
We define bilinear systems \( \langle Z, Z^* \rangle \) and \( \langle Y, Y^* \rangle \) by introducing

\[
Z^* = C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^m) \times \text{range } E(t)^T \times \text{range } E(\bar{t})^T,
\]

\[
Y^* = C_{EE^+}(I, \mathbb{R}^n) \times \text{range } E(t)^T,
\]

with corresponding bilinear forms

\[
\langle z, z^* \rangle = \langle z, (\eta, \vartheta, \delta, \varepsilon) \rangle = \int \left( \eta^T x + \vartheta^T u \right) dt + \delta^T x(t) + \varepsilon^T x(\bar{t}),
\]

\[
\langle y, y^* \rangle = \langle (g, r), (\lambda, \gamma) \rangle = \int \left( \lambda^T g \right) dt + \gamma^T r,
\]

Then, the bilinear systems \( \langle Z, Z^* \rangle \) and \( \langle Y, Y^* \rangle \) are dual systems.

The unique conjugate operator of \( \mathcal{L} \) is given by \( \mathcal{L}^* : Y^* \to Z^* \) with

\[
\mathcal{L}^*(\lambda, \gamma) = (-E^T \frac{d}{dt} (EE^+ \lambda) - (A + EE^+ \dot{E})^T \lambda, -B^T \lambda, \gamma - E(t)^T \lambda(t), E(\bar{t})^T \lambda(\bar{t})).
\]
Define the operator $\mathcal{T} : \mathbb{V} = \mathbb{Y}^* \times \mathbb{Z} \rightarrow \mathbb{V}^* = \mathbb{Y} \times \mathbb{Z}^*$ as

\[ \mathcal{T}(\Lambda, z) = (\mathcal{L}(z), \mathcal{L}^*(\Lambda) - \mathcal{R}(z)) , \]

with $\mathcal{R} : \mathbb{Z} \rightarrow \mathbb{Z}^*$ given by

\[ \mathcal{R}(z) = (Wx + Su, S^T x + Ru, 0, M_{\text{ex}}(\overline{t})) \]

for $z = (x, u) \in \mathbb{Z}$ and $\Lambda = (\lambda, \gamma) \in \mathbb{Y}^*$.

Then, the necessary optimality conditions (BVP) can be written as

\[ \mathcal{T}(\Lambda, z) = (c, 0) . \]

**Theorem**

The operator $\mathcal{T} : \mathbb{V} \rightarrow \mathbb{V}^*$ is self-conjugate, i.e., we have

\[ \langle \mathcal{T}(v), \tilde{v} \rangle = \langle v, \mathcal{T}(\tilde{v}) \rangle \text{ for all } v, \tilde{v} \in \mathbb{V} . \]
Outline

1. The linear-quadratic optimal control problem
2. The self-conjugate differential-algebraic operator
3. Condensed forms for self-adjoint pairs of matrix functions
4. Self-adjoint DAEs and symplectic flow
Consider the DAE

\[ \dot{z} = Az + f, \]

where the pair \((E, A)\) is self-adjoint, i.e.,

\[ E^T = -E, \quad A^T = A + \dot{E}. \]

Applying a congruence transformation with pointwise nonsingular \(Q \in C^1(I, \mathbb{R}^{n,n})\) gives

\[ Q^T E Q \dot{y} = (Q^T AQ - Q^T \dot{E} Q) y + Q^T f, \quad z = Qy \]

If the pair \((E, A)\) is self-adjoint, then the congruent pair

\[ (\tilde{E}, \tilde{A}) = (Q^T E Q, Q^T AQ - Q^T \dot{E} Q) \]

is also self-adjoint.
Under some constant rank conditions on $I$, there exists a pointwise nonsingular $T \in C^1(I, \mathbb{R}^{n,n})$, such that $T^T \mathcal{E} T$ is of the form

$$
\begin{bmatrix}
\mathcal{E}_{11} & \cdots & \cdots & \mathcal{E}_{1,m} & \mathcal{E}_{1,m+1} & \cdots & \mathcal{E}_{1,m+2} & \cdots & \mathcal{E}_{1,2m} & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\mathcal{E}_{1,m}^T & \cdots & \cdots & \mathcal{E}_{m,m} & \mathcal{E}_{m,m+1} & \cdots & \mathcal{E}_{m,m+2} & \cdots & \mathcal{E}_{m,2m} & 0 \\
\mathcal{E}_{1,m+1}^T & \cdots & \cdots & \mathcal{E}_{m+1,m+1} \\
\mathcal{E}_{1,m+2}^T & \cdots & \cdots & \mathcal{E}_{m+1,m+2} \\
\vdots & \ddots & \ddots & \vdots \\
\mathcal{E}_{1,2m}^T & \cdots & \cdots & 0 \\
\mathcal{E}_{m+1,2m} & \cdots & \cdots & \vdots \\
\mathcal{E}_{m+1,m+1} & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
n_1 \\
\vdots \\
n_m \\
l \\
q_m \\
q_2 \\
q_1
\end{bmatrix}
$$

where $q_1 \geq n_1 \geq q_2 \geq n_2 \geq \ldots \geq q_m \geq n_m$,

$$
\mathcal{E}_{j,j} = -\mathcal{E}_{j,j}^T, \quad j = 1, \ldots, m, \quad \mathcal{E}_{j,2m+1-j} \in C(I, \mathbb{R}^{n_j,q_j+1}), \quad 1 \leq j \leq m-1,
$$

$$
\mathcal{E}_{m+1,m+1} = \begin{bmatrix}
J_p & 0 \\
0 & 0
\end{bmatrix}, \quad J_p := \begin{bmatrix}
0 & I_p \\
-I_p & 0
\end{bmatrix},
$$

and each of the first $m$ block columns of $T^T \mathcal{E} T$ has full column rank, and
\[ T^T A T - T^T \dot{\mathcal{E}} T = \]

\[
\begin{bmatrix}
A_{1,1} & \cdots & \cdots & A_{1,m} \\
\vdots & \ddots & \ddots & \vdots \\
A_{m,1} & \cdots & \cdots & A_{m,m} \\
A_{m+1,1} & \cdots & \cdots & A_{m+1,m} \\
0 & \cdots & 0 & A_{m+2,m} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 \\
A_{2m+1,1}
\end{bmatrix}
\begin{bmatrix}
n_1 \\
\vdots \\
n_m \\
l \\
q_m \\
q_1
\end{bmatrix}
\]

where \( q_1 \geq n_1 \geq q_2 \geq n_2 \geq \ldots \geq q_m \geq n_m \),

\[ A_{i,j} = -\dot{\mathcal{E}}_{i,j}, \quad i = 1, \ldots, m-1, \quad j = m+2, \ldots, 2m+1 - i, \]

\[ A_{j,2m+2-j} = A_{2m+2-j,j}^T = \begin{bmatrix} l_{n_j} & 0 \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{n_j,q_j}), \quad 1 \leq j \leq m, \]

\[ A_{m+1,m+1} = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}, \quad \Sigma_{11} = \Sigma_{11}^T \in C(\mathbb{I}, \mathbb{R}^{2p,2p}), \quad \Sigma_{22} = \Sigma_{22}^T \in C(\mathbb{I}, \mathbb{R}^{l-2p,l-2p}), \]

and \( \Sigma_{22} \) is pointwise nonsingular.
Corollary

Consider a self-adjoint pair \((E, A)\) and suppose that there exists a transformation to the staircase form.

1. The associated DAE is regular \(\iff n_j = q_j \) for all \( j = 1, \ldots, m \).

2. If \( m = 0 \), then the DAE is regular and strangeness-free (d-index 1).

3. If \( m > 0 \), then \( \mu \leq \begin{cases} 2m - 1 & \text{if } 2p = \ell \\ 2m & \text{else} \end{cases} \)
   differentiations are necessary to solve the system.

▷ Algebraic constraints can be specified from the condensed form.
▷ They characterize consistency conditions for boundary values as well as smoothness requirements for the inhomogeneity.
⇒ conditions for unique solvability of the system.
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For the optimal control of ODEs (i.e., $E = I_n$) with invertible $R$, the optimality BVP is associated with a Hamiltonian system

$$\begin{bmatrix}
\dot{x} \\
\dot{\lambda}
\end{bmatrix} = \begin{bmatrix}
A - BR^{-1}S^T & -BR^{-1}B^T \\
SR^{-1}S^T - W & -(A - BR^{-1}S^T)^T
\end{bmatrix} \begin{bmatrix}
x \\
\lambda
\end{bmatrix} + \begin{bmatrix}
f \\
0
\end{bmatrix}$$

that generates a symplectic flow, i.e., the fundamental solution matrix $\Phi$ satisfies

$$\Phi^T J_n \Phi = J_n.$$ 

When $R$ is singular, this reduction to a Hamiltonian system is not possible (even in the ODE case).

Is there nevertheless a symplectic flow describing the dynamical part of a self-adjoint DAE?
For a regular DAE with self-adjoint pair \((E, A)\) in condensed form rearranging the equations gives the system

\[
\begin{align*}
\dot{z}_2 &= A_{11}z_1 + A_{12}z_2 + A_{13}z_3 + \tilde{f}_1, \\
-\dot{z}_1 &= A_{12}^Tz_1 + A_{22}z_2 + A_{23}z_3 + \tilde{f}_2, \\
E_{33}\dot{z}_3 + E_{34}\dot{z}_4 + E_{35}\dot{z}_5 &= A_{13}^Tz_1 + A_{23}^Tz_2 + A_{33}z_3 - \dot{E}_{34}z_4 + (I_r - \dot{E}_{35})z_5 + \tilde{f}_3, \\
-E_{34}^T\dot{z}_3 &= A_{44}z_4 + \tilde{f}_4, \\
-E_{35}^T\dot{z}_3 &= z_3 + \tilde{f}_5.
\end{align*}
\]

with \(A_{44}\) invertible, \(E_{35}\) block upper-triang. with diagonal zero blocks.

Solving the equations for the algebraic constraints by backward substitution (including differentiation) results in the underlying ODE

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = -J_p \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} \\
\bar{A}_{12}^T & \bar{A}_{22}
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix}
\bar{f}_1 \\
\bar{f}_2
\end{bmatrix}.
\]

Thus, the underlying flow is symplectic since the matrix on the right is Hamiltonian!
A global condensed form for self-adjoint pairs

Theorem

Consider a regular DAE with self-adjoint pair \((\mathcal{E}, A)\). Suppose that the underlying flow is symplectic. Then there exists a pointwise nonsingular \(L \in C^1(I, \mathbb{R}^{n,n})\) such that

\[
L^T \mathcal{E} L = \begin{bmatrix}
0 & \mathcal{E}_{12} & 0 \\
-\mathcal{E}_{12}^T & \mathcal{E}_{22} & 0 \\
0 & 0 & \mathcal{E}_{33}
\end{bmatrix}, \quad L^T A L - L^T \mathcal{E} \dot{L} = \begin{bmatrix}
0 & -\dot{\mathcal{E}}_{12} & 0 \\
0 & A_{22} & A_{23} \\
0 & A_{32} & A_{33}
\end{bmatrix},
\]

with \(\mathcal{E}_{12}\) pointwise nonsingular and

\[
L^T f = \begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}, \quad L^{-1} z = \begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix},
\]

Then \(z_2\) is uniquely determined from \(\frac{d}{dt}(E_{12}z_2) = f_1\), and

\[
\mathcal{E}_{33} \dot{z}_3 = A_{32}z_2 + A_{33}z_3 + f_3
\]

has a unique solution \(z_3\) for every sufficiently smooth \(f_3\) and given \(z_2\).
For strangeness-free linear-quadratic optimal control problems the DAE operator associated with the necessary optimality BVP is self-conjugate.

This is not true in general, the BVP may not even be solvable.

The associated pair of matrix functions is self-adjoint.

But not every self-adjoint pair leads to a self-conjugate operator.

Under some constant rank assumptions there exists a staircase form that allows to classify the DAE-BVP.

Self-conjugate BVPs have symplectic flows.

Under slight modifications we can use derivative arrays to obtain a strangeness-free system consisting of an Hamiltonian subsystem together with algebraic constraints.
Thank you very much for your attention.
Challenge Workshop MSO-Tools 2012

"Modeling, Simulation and Optimisation Tools"

September 26-28, 2012 at the Technische Universität Berlin, Germany

http://www3.math.tu-berlin.de/MSOTOOLS2012

Invited Speakers:

Tobias Achterberg (IBM, Germany)
Johan Åkesson (Modelon, Sweden)
Torsten Blochwitz (ITI, Germany)
Pieter J. Mosterman (MathWorks, USA)
Klaus Wolf (Fraunhofer SCAI, Germany)

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