Nested Krylov methods for shifted linear systems

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Several applications require the solution of a sequence of shifted linear systems of the form

$$(A - \omega_k I)\mathbf{x}_k = \mathbf{b}, \tag{1}$$

where $A \in \mathbb{C}^{N\times N}, \mathbf{b} \in \mathbb{C}^N$, and $\{\omega_k\}_{k=1}^n \in \mathbb{C}$ is a sequence of n distinct shifts. For example, shifted linear systems arise in model order reduction as well as in the geophysical exploration of both acoustic and elastic waves.
In our application, we focus on wave propagation through elastic media in a frequency-domain formulation. This formulation has specific advantages when modeling visco-elastic effects. In order to improve the imaging of the earth crust, so-called *full waveform inversion* is computed which is an optimization problem at multiple wave frequencies. Therefore, the grid size must be small enough to describe the wave, which for high frequencies results in very large shifted linear systems of the form (1).
In principle, a sequence of shifted systems (1) can be solved almost at the cost of a single solve using so-called shifted Krylov methods. These methods exploit the property that Krylov subspaces are invariant under arbitrary diagonal shifts ω to the matrix A, i.e.,

$$\mathcal{K}_m(A, \mathbf{b}) = \mathcal{K}_m(A - \omega I, \mathbf{b}), \quad \forall m \in \mathbb{N}, \forall \omega \in \mathbb{C}. \tag{2}$$

However, in practical applications, the preconditioning of (1) is required which in general destroys the shift-invariance property (2). In [1], a polynomial preconditioner that preserves the shift-invariance is suggested. The presented work [2] is a new approach to the iterative solution of (1). We use *nested* Krylov methods that use an inner multi-shift Krylov method as a preconditioner for a flexible outer Krylov iteration. In order to deal with the shift-invariance, our algorithm assumes the inner Krylov method to produce collinear residuals for the shifted systems. In my presentation, I will concentrate on two possible combinations of Krylov methods for the nested framework, namely FOM-FGMRES and IDR-FQMRIDR. Since residuals in multi-shift IDR are not collinear by default, the development of a collinear IDR variant which is suitable as an inner method in the new framework is a second main contribution of our work.
An extension of [2] to shifted systems with multiple right-hand sides $B \equiv [\mathbf{b}_1, ..., \mathbf{b}_\ell], \ell \ll N$, using block Krylov methods is subject to our current research.

References

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