Cyclic Reduction and index reduction/shifting for a second-order probabilistic problem

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I wish to describe a problem that has many similarities with the differential-algebraic and boundary-value problems that appear in mechanics and control theory, although it has a different application background and different involved matrix structures.

Markov-modulated Brownian motion is a probabilistic process used in modelling a variety of real-life phenomena. The model consists in a real-valued stochastic process which evolves under a Brownian motion law whose parameters depend on the state of an underlying (environment) continuous-time Markov chain with n states. Its stationary distribution can be represented as a vector-valued function $f : [0, \infty] \mapsto \mathbb{R}^n_{\geq 0}$ which satisfies the constant-valued differential-algebraic equation

$$\ddot{f}(x)V - \dot{f}(x)D + f(x)Q = 0, \tag{1}$$

where $Q \in \mathbb{R}^{n \times n}$ is the generator matrix of a continuous-time Markov chain (a singular $-M$-matrix), while $V \in \mathbb{R}^{n \times n}_{\geq 0}$ and $D \in \mathbb{R}^{n \times n}$ (with mixed signs) are diagonal. Boundary conditions are at 0 and ∞ (or 0 and $M > 0$, in some problems). A classical approach to solving (1) is identifying the invariant subspace associated to the stable eigenvalues o the matrix polynomial $V\lambda^2 - D\lambda + Q$. For instance, a normwise stable approach based on linearization + generalized Schur form exists in literature [1].

We describe an approach based on Cyclic Reduction, a famous matrix iteration, to solve this problem in a componentwise accurate way, relying on the sign properties of the involved matrices and using a special subtraction-free variant of Gaussian elimination, the *GTH method*. This work extends our previous research [2] on first-order problems (those with $V = 0$, known as *fluid queues*). Some novel features appear for second-order problems:

- Switching to a more general formulation with invariant pairs (instead of a matrix equation) is necessary to ensure the correct signs for subtraction-free methods.
- There is less freedom in the choice of the eigenvalue transformation map, an intermediate step that has some points in common with discretization methods for the solution of ODEs.
- In the cases in which V has zero diagonal entries, postmultiplication by a matrix pencil is used to modify the position of the infinite eigenvalues. This transformation can be interpreted as index reduction via differentiation of some equations; in addition to adjusting the eigenvalue positions, it plays an important role in getting the correct signs to ensure the applicability of the componentwise-accurate methods.

References

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