Block Krylov subspace methods for shifted systems with different right-hand sides

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We present some new techniques for solving a family (or a sequence of families) of linear systems in which the coefficient matrices differ only by a scalar multiple of the identity (shifted systems). Our goal is to develop methods for shifted systems which have fewer restrictions usually associated with such methods (e.g., all residuals needing to be collinear).
The systems are parameterized by i,

$$Ax = b \text{ and } (A + \sigma_i I)x(\sigma_i) = b(\sigma_i), i = 1, 2, \ldots, L, \tag{1}$$

with $A \in \mathbb{C}^{n\times n}$ and $\{\sigma_1, \ldots \sigma_L\} \subset \mathbb{C}$. We can add a new parameter j, indexing a sequence of matrices $\{A_j\} \subset \mathbb{C}^{n\times n}$, and for each j we solve a family of systems

$$A_j x_j = b_j \text{ and } (A_j + \sigma_{i,j} I)x(\sigma_{i,j}) = b(\sigma_{i,j}), i = 1, 2, \ldots, L_j, \text{ where } \{\sigma_{i,j}\}_{i=1}^{L_j} \subset \mathbb{C}. \tag{2}$$

Many methods have been proposed for solving (1) are built upon the invariance of Krylov subspaces under a scalar shift, i.e.,

$$\mathcal{K}_j(A, r_0) = \mathcal{K}_j(A + \sigma I, r_0(\sigma)) \tag{3}$$

which holds as long as the collinearity condition $r_0(\sigma) = \beta_\sigma r_0$ is satisfied. This allows us to generate approximate solution corrections for all linear systems in (1) from the common Krylov subspace. These methods can be quite effective and allow for a great savings in both storage and computational costs. However, building methods on top of the invariance (3) introduces a severe restriction on the class of problems we can treat. Furthermore, we have shown that this restriction also hampers the integration of augmentation methods such as subspace recycling [2] into this setting in order to treat (2); see, [5].
Here we propose a technique which circumvent this problem while still taking advantage of the invariance (3). Block Krylov subspaces are shift invariant just as their single-vector counterparts. Thus by collecting all initial residuals into one block vector, we can generate a block Krylov subspace. Due to shift invariance, we can define block FOM- and GMRES-type projection methods to simultaneously solve all shifted systems. These are not block versions of the shifted FOM method [3] or the shifted GMRES method [1]. These methods are compatible with unrelated right-hand sides and residual collinearity is no longer a requirement at restart. Due to this special manner in which we take advantage of (3), subspace recycling may be integrated into the proposed methods in order to treat (2).

References

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