Asynchronous Optimized Schwarz Methods

F. Magoulés[1], C. Venet[2], and D. Szyld[3]

[1]École Centrale Paris, `frederic.magoules@hotmail.com`
[2]École Centrale Paris
[3]Temple University, `szyld@temple.edu`

Asynchronous methods refer to parallel iterative procedures where each process performs its task without waiting for other processes to be completed, i.e., with whatever information it has locally available and with no synchronizations with other processes. Mathematical models of this computational paradigm were developed in the 1980s and 90s and convergence proofs given; see, e.g., the survey [1] and references therein.
Schwarz iterative methods were originally devised to show existence of solutions of elliptic problems on irregular domains and were revived as numerical methods in the 1980s; see, e.g., [4], and the many references therein. For these Schwarz methods, one may consider the solution of a general problem of the form

$$\begin{cases} \mathcal{L}(u) = f \text{ in } \Omega \\ \mathcal{C}(u) = g \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{L}$ and $\mathcal{C}$ a partial differential operators defined on the domain Ω and its boundary, respectively. This domain is (artificially) split into two or more (possibly overlapping) subdomains, i.e., we have $\Omega = \cup_{i=1,\ldots,p}\Omega_i$. In essence one is introducing new artificial boundary conditions on the interfaces between these subdomains. In the classical formulation, these artificial boundary conditions are of Dirichlet type. Given an initial approximation $u(0)$, the method progresses by solving for $u(n+1)$ the equation (1) restricted to each subdomain Ω_i using as boundary data on $\delta\Omega_i \setminus \delta\Omega$ the values for $u(n)$. This procedure is inherently parallel, since the (approximate) solutions on each subdomain can be performed by a different processor.
Convergence of these iterations can be guaranteed under mild conditions, but it is in general rather slow, comparable to the Block Jacobi or Block Gauss-Seidel methods for linear algebraic systems. Much faster convergence can be achieved by using Robin and mixed boundary conditions on the interfaces. In this way one can optimize the Robin parameter(s) and obtain a very fast method. This technique has been termed optimized Schwarz methods; see, e.g., [2]. See also [3] for an algebraic version of this approach.
In this talk, an asynchronous version of the optimized Schwarz method is presented for the solution of differential equations of the form (1) on a parallel computational environment. In a one-way subdivision of the computational domain, with overlap, the method is shown to converge when the optimal artificial interface conditions are used. Convergence is also proved under very mild conditions on the size of the subdomains, when approximate (non-optimal) interface conditions are utilized. Numerical results are presented on large three-dimensional problems illustrating the efficiency of the proposed asynchronous parallel implementation of the method.

References

[1] Andreas Frommer and Daniel B. Szyld. On asynchronous iterations. *Journal of Computational and Applied Mathematics*, 123:201–216, 2000.

[2] Martin J. Gander. Optimized Schwarz methods. *SIAM Journal on Numerical Analysis*, 44(2):699–731, 2006.

[3] Martin J. Gander, Sébastien Loisel, and Daniel B. Szyld. An optimal block iterative method and preconditioner for banded matrices with applications to PDEs on irregular domains. *SIAM Journal on Matrix Analysis and Applications*, 33:653–680, 2012.

[4] Andrea Toselli and Olof Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Series in Computational Mathematics*. Springer, Berlin, Heidelberg, New York, 2005.