Hadamard Semidifferentials
Semidifferential of Parametrized Minima
with Applications to
Shape and Topological Derivatives

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• Motivated by shape and topological derivatives, we revisit the Hadamard semidifferential, for which a complete semidifferential calculus is available, including the chain rule. The *conical derivative* of Mignot [Contrôle dans les inéquations variationelles elliptiques, J. Funct. Anal., 22 (1976)] is a Hadamard semidifferential. It is also a natural tool for differentiation along trajectories in automatic differentiation.

• For real-valued functions we recall the *generalized directional derivative* (an upper semidifferential) for which some form of differential calculus is restored by going to subdifferentials. Both families of functions contain the continuous convex functions, but they are not contained in one another. The choice is problem dependent, but the Hadamard semidifferential is more convenient in most applications.

• The second object of this lecture is the differentiation of the infimum of parametrized objective functions with respect to the parameters as in Danskin [The theory of max-min, with applications, SIAM J. on Appl. Math. **14** (1966)] who obtained a semidifferential equal to the infimum over the set of minimizers of the one-sided directional derivative with respect to the parameters. Yet, in applications to the topological and shape derivatives of the compliance, examples reveal the possible occurrence of an extra negative term: the so-called *polarization term* in Mechanics.

• For the shape derivative, the associated technique is a change of variable to work on the fixed initial domain; for the topological derivative, it is an extension over the hole created by the topological perturbation of the domain.

• This work has applications to compliance problems and to eigenvalue problems where the first eigenvalue is not simple.

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When the variable at hand is a geometric object in the n-dimensional Euclidean space $\mathbb{R}^n$, it is natural to introduce a space of subsets of some fixed *holdall* $D \subset \mathbb{R}^n$ and to give it an appropriate structure (group, metric) to deal with optimal design/control problems and a framework to do sensitivity analysis (differential calculus). There are several ways to do it mathematically.

For shape variations, choose the variable sets as the images of a fixed set Ω_0 by a group of diffeomorphisms with a metric and a differential structure such as the metric groups introduced by Anna-Maria Micheletti [76] in 1972

$$\mathcal{F}(C_0^k(\mathbb{R}^n;\mathbb{R}^n)) \stackrel{\text{def}}{=} \left\{ F : \mathbb{R}^n \to \mathbb{R}^n \text{ bijective} : F - I \text{ and } F^{-1} - I \in C_0^k(\mathbb{R}^n;\mathbb{R}^n) \right\},$$

where $C_0^k(\mathbb{R}^n;\mathbb{R}^n)$ is the space of C^k mappings of $\mathbb{R}^n$ going to zero at infinity. Obviously, such diffeomorphisms can only induce *changes in the shape* of the set Ω_0.

This group was endowed with a metric that she called *Courant metric*. This approach is not limited to $C_0^k(\mathbb{R}^n;\mathbb{R}^n)$, but extends to other spaces such as the Lipschitzian mappings ([43, Chapter 3]).

Since the tangent space to that metric group $\mathcal{F}(C_0^k(\mathbb{R}^n;\mathbb{R}^n))$ is precisely the linear space $C_0^k(\mathbb{R}^n;\mathbb{R}^n)$, we have a notion of *shape derivative*.

The *velocity method* of Zolésio [108] in 1979 corresponds to a trajectory $t \mapsto T_t$ in the group of diffeomorphisms for which the velocity $V(t) \circ T_t$ at T_t belongs to the tangent space $C_0^k(\mathbb{R}^n;\mathbb{R}^n)$ and $dT_t/dt = V(t) \circ T_t$.

When the variable at hand is a geometric object in the n-dimensional Euclidean space $\mathbb{R}^n$, it is natural to introduce a space of subsets of some fixed *holdall* $D \subset \mathbb{R}^n$ and to give it an appropriate structure (group, metric) to deal with optimal design/control problems and a framework to do sensitivity analysis (differential calculus). There are several ways to do it mathematically.

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Perturb the bounded open domain Ω by a family of diffeomorphisms T_t generated by a smooth velocity field $V(t)$:

$$\boxed{\Omega_t \stackrel{\text{def}}{=} T_t(\Omega)}, \quad T_t(X) \stackrel{\text{def}}{=} x(t;X),\ t \geq 0, \quad \frac{dx}{dt}(t;X) = V(t, x(t;X)),\ x(0;X) = X.$$

Given $f \in H^1(\mathbb{R}^n)$, consider the volume integral and the change of variable T_t

$$J(\Omega_t) = \int_{\Omega_t} f\, dx = \int_\Omega f \circ T_t\, j_t\, dx. \quad j_t = \det DT_t, \qquad DT_t \text{ is the Jacobian matrix,}$$

$$dJ(\Omega; V) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t} = \int_\Omega \nabla f \circ V(0) + f \operatorname{div} V(0)\, dx = \int_\Omega \operatorname{div}(f\, V(0))\, dx.$$

T_t will also be used in integrals involving functions u_t and v_t in $H^1(\Omega_t)$ to obtain an integral over Ω and functions $u^t = u_t \circ T_t$ and $v^t = v_t \circ T_t$ in the fixed space $H^1(\Omega)$:

$$\int_{\Omega_t} \nabla u_t \cdot \nabla v_t - a\, v\, dx = \int_\Omega \left[A(t) \nabla u^t \cdot \nabla v^t - a \circ T_t\, v^t j_t \right] dx \tag{2.1}$$

$$A(t) = j_t\, DT_t^{-1} (DT_t^{-1})^\top, \quad j_t = \det DT_t, \qquad DT_t \text{ is the Jacobian matrix,} \tag{2.2}$$

where $(DT_t^{-1})^\top$ is the transpose of the inverse of DT_t.

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For topological variations the variable sets A are identified with a family of set-parametrized functions such as, for instance, the *characteristic function*

$$\chi_A(x) \stackrel{\text{def}}{=} \begin{Bmatrix} 1, & x \in A \\ 0, & x \notin A \end{Bmatrix}. \tag{2.3}$$

Given a *holdall* D (a Lebesgue measurable subset of $\mathbb{R}^n$), let $\mathcal{P}(D)$ be the σ-algebra of Lebesgue measurable subsets of D and m_n the Lebesgue measure in $\mathbb{R}^n$.

Identify the measurable subsets Ω of the hold-all D with their characteristic functions

$$\Omega \in \mathcal{P}(D) \longleftrightarrow \chi_\Omega \in \mathrm{X}(D) \stackrel{\text{def}}{=} \{f \in L^\infty(D) : f(1-f) = 0 \text{ a.e.}\}, \tag{2.4}$$

where $L^p(D) = L^p(D, \mathrm{m}_n)$. Introduce the *Abelian group structure*

$$A \triangle B \stackrel{\text{def}}{=} (A\backslash B) \cup (B\backslash A), \quad (\chi_A \triangle \chi_B)(x) \stackrel{\text{def}}{=} \chi_{A\triangle B}(x) = |\chi_A(x) - \chi_B(x)|, \tag{2.5}$$

where $A \triangle B$ is the *symmetric difference*, $\chi_\varnothing = 0$ is the neutral element, and χ_A is its own *inverse*. The group $\mathrm{X}(\mathbb{R}^n)$ is a closed subset without interior of $L^\infty(\mathbb{R}^n)$ with the associated metric on equivalence classes of measurable subsets of $\mathbb{R}^n$:

$$\rho([\Omega_2], [\Omega_1]) \stackrel{\text{def}}{=} \|\chi_{\Omega_2} - \chi_{\Omega_1}\|_{L^\infty(\mathbb{R}^n)} = \|\chi_{\Omega_2} \Delta \chi_{\Omega_1}\|_{L^\infty(\mathbb{R}^n)},$$

where the operation Δ is continuous. Hence a complete metric group.

Given a topological vector space Y, we are interested in *functions* of the type

$$\chi \mapsto F(\chi) : \mathrm{X}(D) \to Y. \tag{2.6}$$

The original notion of *topological derivative* by removing a small ball around a point $\bar x$ in an open set Ω is the *set derivative of Lebesgue* (or Lebesgue differentiation theorem) with respect to the *dilatation* of a point $\bar x \in \Omega$. It corresponds to a *delta function* in the tangent space to the group $\mathrm{X}(D)$, that is, a bounded measure.

For instance, consider the volume integral of a function $f \in L^1(D)$

$$\chi_\Omega \mapsto V(\chi_\Omega) \overset{\text{def}}{=} \int_{\mathbb{R}^n} \chi_\Omega\, f\, dm_n = \int_\Omega f\, dx : \mathrm{X}(D) \to \mathbb{R}$$

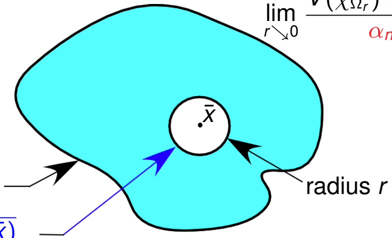
$$E = \{\bar x\}$$

$$E_r = \overline{B_r(\bar x)} \subset \Omega$$

$$\lim_{r \searrow 0} \frac{V(\chi_{\Omega_r}) - V(\chi_\Omega)}{\alpha_n r^n} = \int_{\mathbb{R}^n} \frac{\chi_{\Omega_r} - \chi_\Omega}{\alpha_n r^n}\, f\, dx$$

$$= -\lim_{r \searrow 0} \frac{1}{\alpha_n r^n} \int_{B_r(\bar x)} f\, dx = -f(\bar x)$$

α_n = volume of the unit ball in $\mathbb{R}^n$

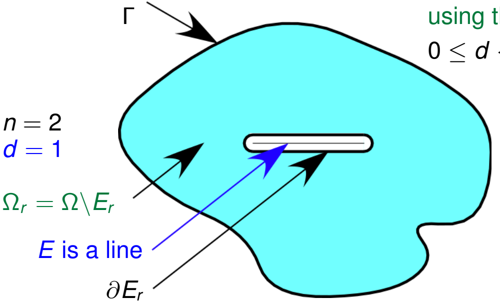


$$\Omega_r = \Omega \backslash \overline{B_r(\bar x)}$$

$$E_r = \overline{B_r(\bar x)}$$

This idea of dilatation extends to curves, surfaces, and d-rectifiable subsets E of $\mathbb{R}^n$, where $0 \le d < n$ is the *dimension* of the perturbing subset E of $\mathbb{R}^n$.

We obtain a semidifferential (one-sided directional derivative) with respect to bounded measures corresponding to points ($d = 0$), curves ($d = 1$), surfaces ($d = 2$), or closed d-rectifiable subsets $E \subset \Omega$ ([27, 28, 31, 32]).



using the d-dimensional Minkowski content

$0 \leq d < n$, E closed, d-rectifiable, $H^d(E) < \infty$

$$t > 0, \quad r = \left(\frac{t}{\alpha_{n-d}}\right)^{1/(n-d)}$$

perturbed set $\Omega_t \overset{\text{def}}{=} \Omega \backslash E_r$

$$\frac{V(\chi_{\Omega_t}) - V(\chi_\Omega)}{t} = \int_{\mathbb{R}^n} \frac{\chi_{\Omega_t} - \chi_\Omega}{t} f\, dx$$

$$= -\frac{1}{\alpha_{n-d}\, r^{n-d}} \int_{E_r} f\, dx$$

$$dV(\chi_\Omega ; \delta_{E,H^d}) = -\int_E f\, dH^d$$

The definition of the topological derivative as a semidifferential was introduced at IFIP 2015 in Sophia Antipolis in "System Modeling and Optimization (CSMO 2015)," L. Bociu, J. A. Desideri and A. Habbal, eds., pp. 230–239, Springer, 2017.

- *Topological derivative: a semidifferential via the Minkowski content*, Journal of Convex Analysis (3) **25** (2018), 957–982.

- *Topological Derivative of State Constrained Objective Functions: a Direct Approach*, SIAM J. on Control and Optim. (1) **60** (2022), 22–47.

- *Topological derivatives via one-sided derivative of parametrized minima and minimax*, Engineering Computations (1) **39** (2022), pp. 34–59.

- *One-sided Derivative of Parametrized Minima for Shape and Topological Derivatives*, SIAM J. Control and Optim., accepted December 2022.

This topological derivative is technically more challenging than the shape derivative. In the literature it is obtained by compound and matched asymptotic expansions Quoting S.A. Nazarov [83],

> *Formulas for increments in the three-dimensional problem, obtained in [85] by the shape optimization tools [98, 43], involve the so-called material derivatives of the energy functionals, which (i.e. the derivatives) are not easily interpreted in natural mechanical terms, and, consequently, one is not able to derive a formula for increments in a simple and clear form. Similar difficulties arise if one calculates the topological derivative of the shape functional. A method for checking the coincidence of formulas obtained by different methods was suggested in [84], but it requires complicated transformations, in particular, multiply repeated integration by parts.*

There is a definite interest in developing this idea of the topological derivative as a semidifferential and direct methods such as the t-derivative and the parametrized minima and minimax formulations for constrained objective functions as an alternative and a complement to compound and matched asymptotic expansions.

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According to Tihomirov [102], “the correct definition of derivative and differential of a function of many variables was given by K. Weierstrass in his lectures in the eighties of the 19th century. These lectures were published in the thirties of our century (20th). ”

Over the years equivalent definitions became available, but the geometrical one of J. Hadamard [62] in 1923 revisited by Fréchet of [57] in 1937 is especially interesting.

Definition

An *admissible trajectory* at $x \in X$ in a topological vector space (TVS) X is a function $h : (-\tau, \tau) \to X$, for some $\tau > 0$, such that

$$h(0) = x \quad \text{and} \quad h'(0) \overset{\text{def}}{=} \lim_{t \to 0} \frac{h(t) - h(0)}{t} \text{ exists in } X, \tag{3.1}$$

where $h'(0)$ is the *tangent* to the trajectory h at $h(0) = x$.

Definition

Let X and Y be topological vector spaces.
A function $f : X \to Y$. is *Hadamard differentiable at* $x \in X$ if there exists a linear function $Df(x) : X \to Y$ such that for each *admissible trajectory* h in X at x,

$$(f \circ h)'(0) \text{ exists and } (f \circ h)'(0) = Df(x)h'(0).$$

All operations of the differential calculus including the chain rule are available.

In 1925 Fréchet extended his 1911 definition[1] in [53] for functions of several variables to functions of functions (functionals).

DEFINITION (FRÉCHET [56] IN 1925)

Let X be a normed space and Y a topological vector space. The function $f : X \to Y$ is *Fréchet differentiable* at $x \in X$ if there exists a *continuous linear mapping* $Df(x) : X \to Y$ such that

$$\lim_{\|v\|\to 0} \frac{f(x+v) - f(x) - Df(x)v}{\|v\|} = 0 \text{ in } Y. \tag{3.2}$$

The linear mapping $v \mapsto Df(x)v : X \to Y$ is the *differential* of f at x.

In finite dimension Hadamard coincides with the Fréchet differential.
But in abstract vector spaces without a norm or a metric, Hadamard's definition is more general as acknowledged by Fréchet [57, Abstract, pp. 233] in 1937:

> *Abstract. The author shows that the definition of the total derivative of Stolz-Young is equivalent to the definition of Mr Hadamard. On the other hand, when the latter is extended to functionals, it becomes more general than the one of the author. Lastly the definition due to M. Paul Lévy, not necessarily verifying the theorem of composite functions, is still more general, but for this very reason, perhaps too general.*

[1] Stolz [99] in 1893, Pierpont [94, 95] in 1905, Young [104, 103] in 1909.

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In 1937 Fréchet [57] also boldly drops the linearity and gives examples of (non-differentiable) homogeneous functions for which the classical differential calculus is preserved. However, this was not sufficient to catch the norm and the convex functions but he was very close. He only had to use semitrajectories over trajectories.

Definition

An *admissible semitrajectory* at x in a topological vector space X is a function $h : [0, \tau) \to X$, for some $\tau > 0$, such that

$$h(0) = x \quad \text{and} \quad h'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{h(t) - h(0)}{t} \text{ exists in } X, \tag{3.3}$$

where $h'(0^+)$ is the *semitangent* to the trajectory h at $h(0) = x$.

Definition

Let X and Y be topological vector spaces. A function $f : X \to Y$. is *Hadamard semidifferentiable at* $x \in X$ if there exists a function $v \mapsto d_H f(x; v) : X \to Y$ such that for each *admissible semitrajectory* h in X at x,

$$(f \circ h)'(0^+) \text{ exists and } (f \circ h)'(0^+) = d_H f(x; h'(0^+)).$$

It can be shown thatt $v \mapsto d_H f(x; v) : X \to Y$ is positively homogeneous and sequentially continuous (continuous in Fréchet spaces).

Fréchet [57, p. 239] gives the following example.

$$f(x,y) = x\sqrt{\frac{x^2}{x^2+y^2}} \quad \text{for } (x,y) \neq (0,0) \quad \text{with } f(0,0)=0 \tag{3.4}$$

Indeed, it is readily checked that at $(0,0)$ with $h(t) = (x(t), y(t))$. $h'(0) \neq (0,0)$,

$$\frac{x(t)\sqrt{\frac{x(t)^2}{x(t)^2+y(t)^2}} - 0}{t} = \frac{x(t)-x(0)}{t}\sqrt{\frac{\left(\frac{x(t)-x(0)}{t}\right)^2}{\left(\frac{x(t)-x(0)}{t}\right)^2 + \left(\frac{y(t)-y(0)}{t}\right)^2}}$$

$$\Rightarrow \lim_{t\to 0} \frac{f(x(t),y(t)) - f(x(0),y0))}{t} = x'(0)\sqrt{\frac{(x'(0))^2}{(x'(0))^2+(y'(0))^2}}$$

$$\Rightarrow d_H f((0,0);(v_1,v_2) = \begin{cases} v_1\sqrt{\frac{v_1^2}{v_1^2+v_2^2}} & (v_1,v_2)\neq(0,0) \\ 0, & (v_1,v_2)=(0,0) \end{cases} = f(v_1,v_2); \mathbb{R}^2 \to \mathbb{R}$$

which is not linear in (v_1, v_2).

This definition was crticized by Paul Lévy.

Far from discrediting this new notion, this example shows that such functions exist.

By using $h'(0)$ and $(f \circ h)'(0)$ rather than $h'(0^+)$ and $(f \circ h)'(0^+)$, Fréchet was losing some Hadamard semidifferentiable functions such as the Euclidean norm[2] $n(x) = \|x\|$ on $\mathbb{R}^n$ at $x = 0$ since the differential quotient

$$\frac{n(h(t)) - n(h(0))}{t} = \frac{\|h(t)\| - \|0\|}{t} = \frac{|t|}{t} \left\| \frac{h(t) - 0}{t} \right\| \quad \text{diverges as } t \to 0. \tag{3.5}$$

It is really necessary that t be positive ($t = |t|$) to get the convergence of the limit of the differential quotient

$$\lim_{t \searrow 0} \left\| \frac{h(t) - 0}{t} \right\| = \|h'(0^+)\| \quad \Rightarrow d_H n(x; v) = \begin{cases} \dfrac{x}{\|x\|} \cdot v, & x \neq 0 \\ \|v\|, & x = 0. \end{cases}$$

Yet, it is quite remarkable that, up to the use of the right-hand side derivatives $h'(0^+)$ and $(f \circ h)(0^+)$ rather than the derivatives $h'(0)$ and $(f \circ h)'(0)$, Fréchet introduced a class of nondifferentiable functions verifying the theorem of composite functions.

[2] The inner product in $\mathbb{R}^n$ is denoted $x \cdot y$ and the norm $\|x\| = \sqrt{x \cdot x}$.

This Hadamard semidifferentiability preserves all the operations of the differential calculus including the chain rule and more. For instance, for $f_1,\ f_2; X \to Y$

$$d_H(\alpha f_1 + \beta f_2)(x; v) = \alpha d_H f_1(x; v) + \beta d_H f_2(x; v), \quad \alpha, \beta \in \mathbb{R}, \tag{3.6}$$

for $g : X \to Y$ and $f : Y \to Z$

$$d_H(f \circ g)(x; v) = d_H f(g(x); d_H g(x; v)). \tag{3.7}$$

Moreover, additional operations such as the lower and upper envelops of a finite family of real-valued functions are available: for $f_i : X \to \mathbb{R}$, $i = 1, \dots m$,

$$d_H\left(\max_{1\le i\le m} f_i\right)(x; v) = \max_{i\in I(x)} d_H f_i(x; v), \quad I(x) = \{i : f_i(x) = \max_{1\le j\le m} f_j(x)\} \tag{3.8}$$

$$d_H\left(\min_{1\le i\le m} f_i\right)(x; v) = \min_{i\in J(x)} d_H f_i(x; v), \quad J(x) = \{i : f_i(x) = \min_{1\le j\le m} f_j(x)\}. \tag{3.9}$$

This includes the functions $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \min\{f(x), 0\}$.

All continuous convex (resp. concave) functions on X are Hadamard semidifferentiable in the interior of their domain.

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The Hadamard semidifferential is the natural choice for functions on a subset A of X.

For a closed sufficiently smooth embedded submanifold A of $X = \mathbb{R}^n$ of dimension $d < n$, $\overline{\mathbb{R}^n \setminus A} = \mathbb{R}^n$, $A = \partial A$, and the smoothness insures that, at each point of A, the tangent space is a d-dimensional linear subspace. This is illustrated below in Figure 1

$h'(0) \overset{\text{def}}{=} \lim_{t\to 0} \frac{h(t)-x}{t}$ exists

$T_A(x) = \mathbb{R}$ is a linear subspace of $\mathbb{R}^2$

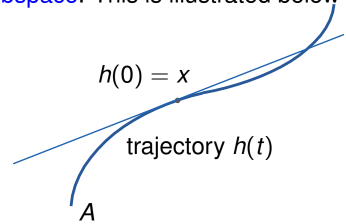

FIGURE 1: Tangent $h'(0)$ to the trajectory h in A at the point $h(0) = x$.

for a smooth curve A in $\mathbb{R}^2$.

But, the linearity of $T_A(x)$ puts a severe restriction on the choice of sets A. For instance, the requirement that $T_A(x)$ be linear rules out a curve in $\mathbb{R}^2$ with a kink at x as shown in the Figure 2.

$h'(0^+) \overset{\text{def}}{=} \lim_{t\searrow 0} \frac{h(t)-x}{t}$ exists

$T_A(x)$ is a non-convex cone in 0

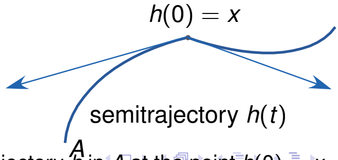

FIGURE 2: Half-tangent $h'(0^+)$ to the semitrajectory h in A at the point $h(0) = x$.

DEFINITION ([32])

Let $A \neq \varnothing$ be a subset of a topological vector space X.
An *admissible semitrajectory* at $x \in A$ in A is a function $h : [0, \tau) \to A$ such that

$$h(0) = x \quad \text{and} \quad h'(0^+) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{h(t) - h(0)}{t} \text{ exists in } X, \tag{3.10}$$

where $h'(0^+)$ is the *semitangent* to the trajectory h in A at $h(0) = x$.

DEFINITION (AUBIN-FRANKOWSKA [6, DFN. 4.1.5, PP. 127–128 AND P. 161], [32])

Let $A \neq \varnothing$ be a subset of a topological vector space X.
The *adjacent tangent cone* to A at $x \in A$ is defined as

$$T_A^\flat(x) \stackrel{\text{def}}{=} \left\{ v \in X : \forall \{t_n \searrow 0\}, \exists \{x_n\} \subset A \text{ such that } \lim_{n\to\infty} \frac{x_n - x}{t_n} = v \right\}.$$

For $x \in \text{int}\, A$, $T_A^\flat(x) = X$. For $x \in \partial A$, the relevant tangent cone to A is $T_A^\flat(x)$.

THEOREM ([32])

Let $A \neq \varnothing$ be a subset of a topological vector space X.

$$\forall x \in A. \quad T_A^\flat(x) = \left\{ h'(0^+) : h \text{ an admissible semitrajectory in } A \text{ at } x \right\}. \tag{3.11}$$

We now have all the elements to extend the definition of the Hadamard semidifferential to a subset A of a TVS X.

DEFINITION ([32])

Let X and Y be topological vector spaces, A, $\varnothing \neq A \subset X$, and $f : A \to Y$.

(i) f is *Hadamard semidifferentiable at* $x \in A$ if there exists a function $v \mapsto d_H f(x; v) : T_A^\flat(x) \to Y$ such that for all *admissible semitrajectories* h in A at x

$$(f \circ h)'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{f(h(t)) - f(h(0))}{t} = d_H f(x; h'(0^+)) \tag{3.12}$$

(ii) f is *Hadamard differentiable at* $x \in A$ if f is Hadamard semidifferentiable at $x \in A$, $T_A^\flat(x)$ is a linear subspace, and the function $v \mapsto d_H f(x; v) : T_A^\flat(x) \to Y$ is linear in which case it will be denoted $Df(x)$. □

REMARK

(i) The *conical derivative* of Mignot [77, Dfn. 2.1 and Prop, 2.3, pp.141–142] in 1976 [Contrôle dans les inéquations variationelles elliptiques] is a Hadamard semidifferential.

(ii) By its very definition, the Hadamard semidifferentiabilty is differentiation along trajectories as in *automatic differentiation* (see, for instance, the paper of J. Bolte and E. Pauwels [9] in 2021 [Conservative set valued fields, automatic differentiation, stochastic gradient methods and deep learning]).

THEOREM ([32])

Let X and Y be topological vector spaces and A, $\varnothing \neq A \subset X$.

(i) *If $f : A \to Y$ is Hadamard semidifferentiable at $x \in A$, then the mapping*

$$v \mapsto d_H f(x;v) : T_A^\flat(x) \to T_{f(A)}^\flat(f(x)) \subset Y \tag{3.13}$$

is sequentially continuous for the induced topologies.

(ii) *If $f_1 : A \to Y$ and $f_2 : A \to Y$ are Hadamard semidifferentiable at $x \in A$, then for all α and β in $\mathbb{R}$,*

$$\forall v \in T_A^\flat(x), \quad d_H(\alpha f_1 + \beta f_2)(x;v) = \alpha\, d_H f_1(x;v) + \beta\, d_H f_2(x;v), \tag{3.14}$$

and $\alpha f_1 + \beta f_2$ is Hadamard semidifferentiable at x.

(iii) *(Chain rule) Let X, Y, Z be topological vector spaces, $g : A \subset X \to Y$, and $f : g(A) \to Z$ be functions such as g is Hadamard semidifferentiable at x and f is Hadamard semidifferentiable at $g(x)$ in $g(A)$. Then $d_H g(x;v) \in T_{g(A)}^\flat(x)$, $f \circ g$ is Hadamard semidifferentiable at x, and*

$$\forall v \in T_A^\flat(x), \quad d_H(f \circ g)(x;v) = d_H f(g(x); d_H g(x;v)). \tag{3.15}$$

We obtain notions of semidifferential and differential without introducing coordinate spaces, charts, local bases, or Christoffel symbols.

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We begin with the standard first order necessary condition for a local minimum.

Theorem ([33])

Let X be a topological vector space, $A \neq \varnothing$ be a subset of X, and $f : A \to \mathbb{R}$ be Hadamard semidifferentiable at $x \in A$.

(i) *If $x \in A$ is a local minimizer of f with respect to A, then*

$$d_H f(x; v) \geq 0 \text{ for all } v \in T_A^\flat(x), \tag{3.16}$$

where $T_A^\flat(x)$ is the adjacent tangent cone.

(ii) *Il A is convex and $x \in A$ is a minimizer of f with respect to A, then*

$$d_H f(x; y - x) \geq 0 \text{ for all } y \in A. \tag{3.17}$$

If, in addition, f is convex, condition (3.17) *is necessary and sufficient.*

M. C. Delfour, Hadamard Semidifferential of Functions on an Unstructured Subset of a TVS, J. Pure and Applied Functional Analysis 5, no. 5, (2020), 1039-1072.

M. C. Delfour, Hadamard Semidifferential, Oriented Distance Function, and some Applications, Communications on Pure and Applied Analysis, 21, no. 6 (2022), 1917-1951. doi:10.3934/cpaa.2021076

M. C. Delfour, Introduction to Optimization and Hadamard Semidifferential Calculus, 2nd edition, MOS-SIAM Series, Phil., USA, 2012. (thanks to Keneth Lange, UCLA)

Mossino and Zolésio [81] in 1977 and Zolésio [107, 108] in 1979 considered the infimum of the following non-differentiable convex continuous functional on $H_0^1(\Omega)$

$$f(v) \stackrel{\text{def}}{=} \int_\Omega \left(\|\nabla v(x)\|^2 + |\Omega|\, v(x) \right) dx + \int_\Omega \int_\Omega [v(x) - v(y)]^+ \, dx\, dy, \tag{3.18}$$

where $[y]^+ = \max\{y, 0\}$.

It provided a direct way to get the Grad-Mercier equation in Plasma Physics.

THEOREM (MOSSINO-ZOLÉSIO [81] AND ZOLÉSIO [107, 108])

Assume that Ω is a bounded open domain with locally Lipschitzian boundary Γ and that f is given by (3.18)*.*

(i) *There exists a unique minimizer $u \in H_0^1(\Omega)$*

(ii) *u is the solution in $H_0^1(\Omega) \cap H^2(\Omega)$ of the following (non-local) system*

$$\boxed{-\Delta u + \beta_-(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma,} \quad \text{meas}\left(\{y \in \Omega : u(x) = u(y)\}\right) = 0, \tag{3.19}$$

where

$$\beta_-(u)(x) = \text{meas}\left(\{y \in \Omega : u(x) > u(y)\}\right). \tag{3.20}$$

It says that the variational solution u is not constant on any subset of Ω of positive measure and is the unique solution of the first equation (3.19) with that property.

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In 1978 Penot [92] introduces the following stronger definition

DEFINITION (PENOT [92, P. 250], 1978)

Let X and Y be topological vector spaces. A function $f : X \to Y$ is *M-semidifferentiable*[a] *at* $x \in X$ if

$$\forall v \in X, \quad d_M f(x; v) \stackrel{\text{def}}{=} \lim_{\substack{w \to v \\ t \searrow 0}} \frac{f(x + tw) - f(x)}{t} \text{ exists in } Y. \tag{3.21}$$

[a] A. D. Michal [71, 72] in 1938 and 1939.

If $f : X \to Y$ is M-semidifferentiable at x, it is Hadamard semidifferentiable at x.

The next theorem connects continuity and semidifferentiability for a convex function.

THEOREM

Let X be a locally convex topological vector space, $f : \text{dom}\, f \to \mathbb{R}$ a convex function, and x a point in the interior of its domain $\text{dom}\, f$.

(i) *If f is continuous.at x, then f is M-semidifferentiable at x.*

(ii) *If f is sequentially continuous at x, then f is Hadamard semidifferentiable at x.*

(iii) *If X is a Fréchet space, then f is continuous at x if and only if f is Hadamard semidifferentiable at x.*

Theorem (normed vector spaces)

Let X and Y be normed vector spaces, $f : X \to Y$ a function, and $x \in X$. The function f is M-semidifferentiable at x if and only if it is Hadamard semidifferentiable at x, that is, $d_M f(x; v) = d_H f(x; v)$.

Definition (Lipschitz functions)

Let X and Y be normed spaces. A function $f : X \to Y$ is *Lipschitz continuous* at $x \in X$ if there exists a constant $c(x) > 0$ and a ball $B_r(x)$ of radius $r > 0$ such that

$$\forall y, z \in B_r(x), \quad \|f(y) - f(z)\|_Y \leq c(x)\, \|y - z\|_X. \tag{3.22}$$

Theorem

Let X and Y be normed vector spaces, $f : X \to Y$ a function, and $x \in X$. If $f : X \to Y$ is Lipschitz at x, then

(i) *f is Hadamard semidifferentiable at x if and only if*

$$\forall v \in X, \quad \lim_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} \text{ exists.} \tag{3.23}$$

(ii) *In particular, if f is convex and x is an interior point of its domain $\text{dom}\, f$, then (3.23) is verified and f is Hadamard semidifferentiable at x.*

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To complete this section, we quote the definition of strict differentiability introduced by the school of Bourbaki in the fifties, which is strictly stronger than the M-, Hadamard, and Fréchet differentiabilities.

DEFINITION (CLARKE [19, P. 30–31])

Given two Banach spaces X and Y, a function $f : X \to Y$ is *strictly differentiable* at x if there exists a continuous linear function $Df(x) : X \to Y$ such that

$$\forall v \in X, \lim_{\substack{t \searrow 0 \\ y \to x}} \frac{f(y+tv) - f(y)}{t} = Df(x)v. \tag{3.24}$$

- *According to [19, Prop. 2.2.1. p. 31] such a function is Lipschitz continuous at x.*

For real-valued functions $f : X \to \mathbb{R}$ Lipschitz continuous at $x \in X$, lower and upper notions of Gateaux, M-, and strict differentiabilty can be introduced by replacing the limit by the $\liminf$ or the $\limsup$. They are called upper and lower semidifferentials in the terminology of Cannarsa and Sinestrari [14].

Upper and lower semidifferentials of locally Lipschitz functions are more general, but the basic operations of the differential calculus are lost and one resorts to the notion of *subdifferential* and the tools of set-valued analysis to restore some form of calculus.

This is a disadvantage over the Hadamard semidifferential.

$$\underline{d}f(x;v) \overset{\text{def}}{=} \liminf_{t\searrow 0} \frac{f(x+tv)-f(x)}{t}$$

lower Gateaux semidifferential at x in the direction v

$$\overline{d}f(x;v) \overset{\text{def}}{=} \limsup_{t\searrow 0} \frac{f(x+tv)-f(x)}{t}$$

upper Gateaux semidifferential at x in the direction v

$$\underline{d}_M f(x;v) \overset{\text{def}}{=} \liminf_{\substack{t\searrow 0\\ w\to v}} \frac{f(x+tw)-f(x)}{t}$$

lower M-semidifferential at x in the direction v

$$\overline{d}_M f(x;v) \overset{\text{def}}{=} \limsup_{\substack{t\searrow 0\\ w\to v}} \frac{f(x+tw)-f(x)}{t}$$

upper M-semidifferential at x in the direction v.

$$\underline{d}_C f(x;v) \overset{\text{def}}{=} \liminf_{\substack{t\searrow 0\\ y\to x}} \frac{f(y+tv)-f(y)}{t}$$

Clarke lower semidifferential at x in the direction v

$$\overline{d}_C f(x;v) \overset{\text{def}}{=} \limsup_{\substack{t\searrow 0\\ y\to x}} \frac{f(y+tv)-f(y)}{t}$$

Clarke upper semidifferential at x in the direction v

The upper notion of strict differentiability $\overline{d}_C f(x;v)$ corresponds to the *upper semidifferential* developed by Clarke [18] in 1973 under the name *generalized directional derivative*.

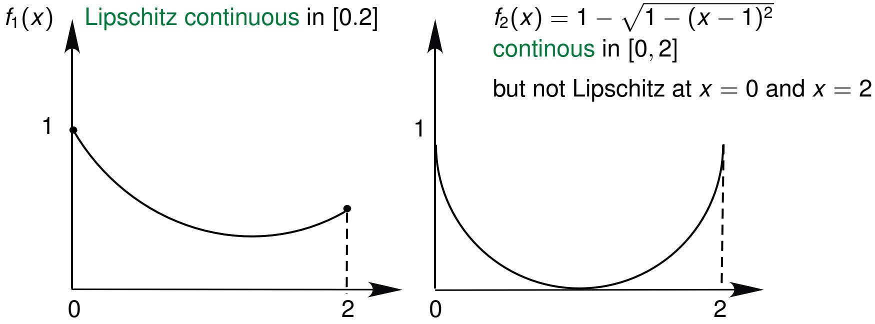
For a convex function f at a point x in the interior of its domain dom f (see, for instance, [19, Prop. 2.2.7, p. 36]).

$$\forall v, \quad \overline{d}_C f(x;v) = df(x;v) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{f(x+tv)-f(x)}{t}.$$

Hence, from our previous considerations,

$$\boxed{\forall v, \quad \overline{d}_C f(x;v) = df(x;v) = d_H f(x;v).}$$

- What is happening at boundary points of a closed convex U?



Note that for a concave function we have $\underline{d}_C f(x;v) = d_H f(x;v)$ for all v.

For the Lipschitz continuous function f_1 on $U = [0,2]$

$$\forall v \in T_x^\flat(U), \quad \overline{d}_C f(x;v) = d_H f(x;v).$$

For the continuous convex function $f_2 : [0,1] \to \mathbb{R}$

$$f_2(x) = 1 - \sqrt{1-(x-1)^2} \tag{3.25}$$

choose an admissible semitrajectory $h : [0,1)$ such that

$$h(0) = 0 \text{ and } h'(0^+) = 1. \tag{3.26}$$

Then for $t > 0$ and $h(t)/t \to 1$

$$\frac{f_2(h(t)) - f_2(h(0))}{t} = -\frac{\sqrt{1-(h(t)-1)^2}}{t} = -\sqrt{\frac{2}{t}\frac{h(t)}{t} - \left(\frac{h(t)}{t}\right)^2} \to -\infty.$$

Moreover, setting $y \searrow 0$ and $t \searrow 0$ in the *strict differential quotient*

$$\frac{f_2(y+t) - f_2(y)}{t} \to -\infty \quad \Rightarrow \boxed{\overline{d}_C f(x;v) = -\infty = d_H f(x;v).} \tag{3.27}$$

Therefore, if we allow in the definition of $d_H f(x;v)$ the value $-\infty$, f_2 is Hadamard semidifferentiable at $x = 0$ for directions in the cone $T_0^\flat([0,2]) = [0,\infty)$.

If a convex function $f : U \to \mathbb{R}$ is continuous on a closed convex subset U

$$\forall x \in \partial U, \forall v \in \mathbb{R}^+(U-x), \quad \boxed{\overline{d}_C f(x;v) = d_H f(x;v),}$$

where this semidifferential can be $-\infty$ as for the example of the function f_2.

In the next two slides, we give two examples:

(i) a function which is Hadamard semidifferentiable at 0, but is not Lipschitz in any neighborhood of $x = 0$;

(ii) a Lipschitz function f which is not Hadamard semidifferentiable at 0.

They shows that
the Hadamard semififferentiable functions are not contained in the Lipschitzian functions with a generalized directional derivative
and that
the Lipschitzian functions with a generalized directional derivative are not contained in the Hadamard semififferentiable functions.

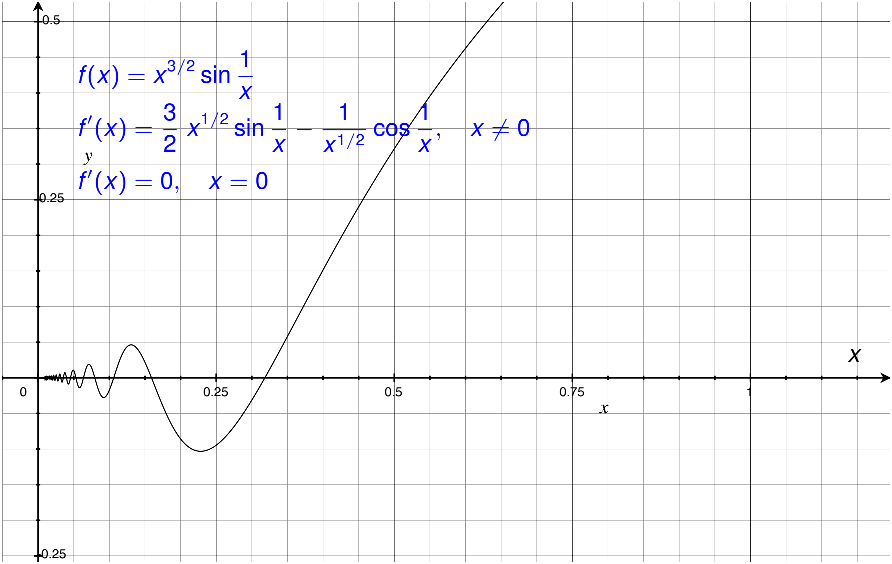
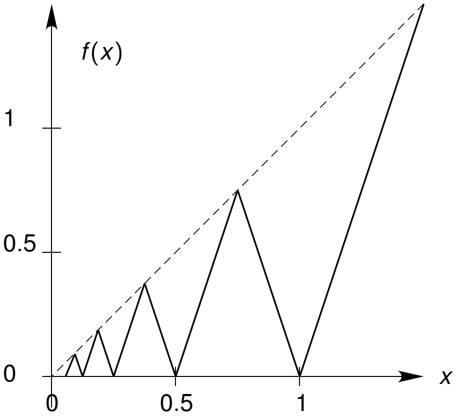


FIGURE 3: A function which is Hadamard semidifferentiable at 0 which is not Lipschitz in any neighborhood of $x = 0$.



$$\frac{1}{2^{(n+1)}} < \frac{3}{2^{n+2}} < \frac{1}{2^n}$$

$$f(x) = 3\left[\frac{1}{2^n} - x\right], \qquad \frac{3}{2^{(n+2)}} < x \leq \frac{1}{2^n}$$
$$f(x) = 3\left[x - \frac{1}{2^{n+1}}\right], \qquad \frac{1}{2^{(n+1)}} < x \leq \frac{3}{2^{n+2}}$$
$$n = -1, 0, 1, 2, \ldots$$
$$f(x) = 0, \quad x \leq 0$$

FIGURE 4: A Lipschitz function f which is not Hadamard semidifferentiable at 0.

For $v = 1$

$$\liminf_{t \searrow 0} \frac{f(0 + tv) - f(0)}{t} = 0, \quad \limsup_{t \searrow 0} \frac{f(0 + tv) - f(0)}{t} = 1. \tag{3.28}$$

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7 REFERENCES

Whether dealing with differentiation in a function space or with shape or topological derivatives, the problem can be put in the following general form:

$$g(t) \stackrel{\text{def}}{=} \inf_{x\in X} G(t,x), \quad (t,x)\mapsto G(t,x) : [0,\tau)\times X \to \mathbb{R}, \tau>0, \tag{4.1}$$

$$\text{to find and characterize } dg(0) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{g(t)-g(0)}{t}. \tag{4.2}$$

Let $X(t) \stackrel{\text{def}}{=} \{x^t \in X : g(t) = G(t,x^t)\}$ be the set of minimizers at $t \geq 0$.

Even if, in many cases, $dg(0)$ is simply given in terms of the *one-sided derivative* of $G(t,x)$ with respect to t (in the sequel we shall use the terminology *t-derivative*)

$$dg(0) = \inf_{x^0\in X(0)} d_tG(0,x^0), \quad \text{where } d_tG(0,x^0) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{G(t,x^0)-G(0,x^0)}{t}, \tag{4.3}$$

there are examples where an *extra negative term* occurs.

This extra term is known as the *polarization term* in the literature on the topological derivative which is often obtained by resorting to compound and matched asymptotic expansions (see, for instance, Sokołowski and Zòchowski [97], Nazarov and Sokołowski [84], [3]), [83], [86], [15], . . .)

In general, those methods are global and do not separate the computation of the extra term from the one of the t-derivative of $G(t,x^0)$. For instance, see the joint use of Fenchel duality and Gamma-convergence techniques by Bouchité, Fragala, and Lucardes. [13] to obtain the shape derivative of minima of integral functionals (see also Ngom, Faye, and Seck [87] for minimax of Lagrangian).

Danskin [22] in 1966 gives several simple examples in which the function g is not differentiable even if G is very smooth. This type of nondifferentiability is closely related to the fact that the set of minimizers $Y(x)$ is not a singleton as illustrated in his example of the *seesaw problem* ([22, p. 643]).

height of the point (x, y): $G(x, y) = y \sin x$

- J_Y minimizes the height over $Y \stackrel{\text{def}}{=} \{y \in \mathbb{R} : |y| \leq 1\}$
  $g(x) = \min_{|y|\leq 1} G(x, y) = -|\sin x|$

- J_X maximizes $g(x)$ over $X \stackrel{\text{def}}{=} \{x \in \mathbb{R} : |x| \leq \pi/2\}$

$\max_{|x|\leq \pi/2} \min_{|y|\leq 1}(y \sin x) = \max_{|x|\leq \pi/2} g(x) = 0$

where the maximum is reached at $x = 0$

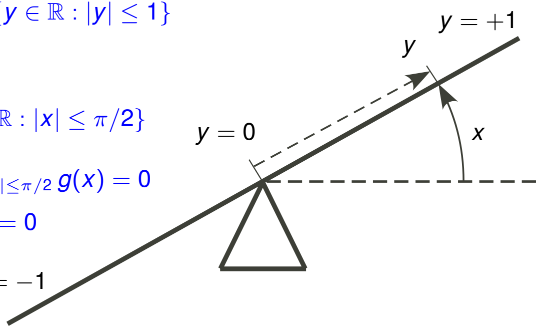

FIGURE 5: The seesaw problem of Danskin where $(x, y) \in [-\pi/2, \pi/2] \times [-1, 1]$.

Player J_X chooses the angle $x \in X$ (see Figure 5) of the seesaw, player J_Y chooses any point y between the extremities -1 and $+1$. Consider the function

$$g(x) = \inf_{y\in Y} G(x,y), \quad G(x,y) \overset{\text{def}}{=} y \sin x, \quad Y \overset{\text{def}}{=} \{y \in \mathbb{R} : |y| \le 1\}, \tag{4.4}$$

It is readily seen that

$$g(x) = \min_{|y|\le 1} (y \sin x) = -|\sin x|, \quad \text{set of minimizers } Y(x) = \begin{cases} \left\{-\dfrac{\sin x}{|\sin x|}\right\}, & x \neq 0, \\ Y, & x = 0. \end{cases}$$

The directional derivative of $G(x,y)$ with respect to x in the direction v

$$d_x G(x,y;v) = (y \cos x)v \tag{4.5}$$

and the directional derivative of $g(x)$ with respect to x in the direction v

$$dg(x;v) = \inf_{y\in Y(x)} (y \cos x)v = \begin{cases} -\dfrac{\sin x \cos x}{|\sin x|} v, & x \neq 0, \\ \inf_{|y|\le 1} y\, v = -|v|, & x = 0. \end{cases} \tag{4.6}$$

The function $g(x)$ is not differentiable at $x = 0$, where the maximum of $g(x)$ occurs.

It is neither convex nor concave. The nondifferentiability at $x = 0$ arises from the fact that the set $Y(x)$ of minimizers of $G(x,y)$ is not a singleton at $x = 0$.

Yet, the function g is Hadamard semidifferentiable.

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Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with Lipschitz boundary Γ and let A be a symmetric $n \times n$ matrix such that

$$\exists \alpha > 0, \; \forall x \in \mathbb{R}^n, \quad Ax \cdot x \geq \alpha \, \|x\|^2. \tag{4.7}$$

The first eigenvalue can be obtained via the *Rayleigh quotient*

$$\lambda(\Omega, A) \overset{\text{def}}{=} \inf_{v \in H_0^1(\Omega),\, v \neq 0} F(\Omega, A; v), \quad F(\Omega, A; v) \overset{\text{def}}{=} \frac{\int_\Omega A\nabla v \cdot \nabla v \, dx}{\int_\Omega |v|^2 \, dx}.$$

The corresponding eigenspace is

$$E(\Omega, A) \overset{\text{def}}{=} \left\{ v \in H_0^1(\Omega) : -\text{div}\,(A\nabla v) = \lambda(\Omega, A) v \right\}.$$

Given a symmetric $n \times n$ matrix B, we are interested in the limit

$$d\lambda(\Omega, A; B) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{\lambda(\Omega, A + tB) - \lambda(\Omega, A)}{t} = \inf_{\substack{v \in E(\Omega, A) \\ v \neq 0}} \frac{\int_\Omega B\nabla v \cdot \nabla v \, dx}{\int_\Omega |v|^2 \, dx}.$$

This problem is defined on the fixed space $X = H_0^1(\Omega)$ for the function

$$v \mapsto G(t, v) \overset{\text{def}}{=} F(\Omega, A + tB; v) : X \to \mathbb{R},$$

$$g(t) \overset{\text{def}}{=} \inf_{v \in X} G(t, v), \quad X(t) \overset{\text{def}}{=} \{u \in X : G(t, u) = g(t)\},$$

$$\Rightarrow \; d\lambda(\Omega, A; B) = dg(0) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{g(t) - g(0)}{t}.$$

Another example is the first eigenvalue of the bi-Laplacian which is not simple.

The first eigenvalue $\lambda(\Omega, A)$ can also be obtained via the Auchmuty's dual principle as follows

$$\left.\begin{aligned} \mu(\Omega, A) &\overset{\text{def}}{=} \inf_{v\in H_0^1(\Omega)} F(\Omega, A; v), \\ F(\Omega, A; v) &\overset{\text{def}}{=} \frac{1}{2}\int_\Omega A\nabla v\cdot\nabla v\, dx - \left[\int_\Omega |v|^2\, dx\right]^{1/2} \end{aligned}\right| \quad \lambda(\Omega, A) = -\frac{1}{2\mu(\Omega, A)}. \tag{4.8}$$

The main advantage of $\mu(\Omega, A)$ is that the minimization is over the linear space $H_0^1(\Omega)$.

It is shown in [59, Chapter 9, sec 2.3.3, pp. 203–205] that this relation holds between the infima of the quotient of two symmetric bilinear forms defined over a Hilbert space X. For $0 \le t \le \tau$,

$$\lambda(t) \overset{\text{def}}{=} \inf_{0\neq v\in X} f(t, v), \quad f(t, v) \overset{\text{def}}{=} \frac{a(t, v, v)}{b(t, v, v)}, \quad 0 \neq v \in X, \tag{4.9}$$

$$df(t, u; v) = \frac{2}{b(t, u, u)}\left[a(t, u, v) - f(t, u)\, b(t, u, v)\right], \; u \neq 0, \tag{4.10}$$

under the assumption that $b(t, v, v) \ge 0$ and $b(t, v, v) = 0$ implies $v = 0$ in X.

The corresponding Auchmuty Dual Problem is

$$\mu(t) \overset{\text{def}}{=} \inf_{v\in X} g(t, v), \quad m(t, v) \overset{\text{def}}{=} \frac{1}{2}a(t, v, v) - b(t, v, v)^{1/2}, \quad \lambda(t) = -\frac{1}{2\mu(t)}, \tag{4.11}$$

$$dm(t, u; v) = a(t, u, v) - \frac{1}{b(t, u, u)^{1/2}}\, b(t, u, v), \; u \neq 0. \tag{4.12}$$

Let $\Omega \subset \mathbb{R}^n$ be bounded open with smooth boundary Γ and A a symmetric $n \times n$ matrix verifying (4.7). Given $f \in L^2(\mathbb{R}^n)$, let $u \in H_0^1(\Omega)$ be the solution of

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma. \tag{4.13}$$

The *compliance* is defined as the *work of the applied forces*

$$J(\Omega, A) \stackrel{\text{def}}{=} -\int_\Omega f\, u\, dx. \tag{4.14}$$

The function $u \in H_0^1(\Omega)$ is the minimizing element of the *energy functional*

$$E(\Omega, A; u) = \inf_{v \in H_0^1(\Omega)} E(\Omega, A; v), \quad E(\Omega, A; v) \stackrel{\text{def}}{=} \int_\Omega A\nabla v \cdot \nabla v - 2f\, v\, dx, \tag{4.15}$$

$$\Rightarrow \exists u \in H_0^1(\Omega) \text{ such that } \forall v \in H_0^1(\Omega), \quad \int_\Omega A\nabla u \cdot \nabla v - f\, v\, dx = 0. \tag{4.16}$$

$$\Rightarrow J(\Omega, A) = -\int_\Omega A\nabla u \cdot \nabla u\, dx = \inf_{v \in H_0^1(\Omega)} E(\Omega, A; v). \tag{4.17}$$

If B is a symmetrical $n \times n$ matrix and $t > 0$, we are interested in computing

$$dJ(\Omega, A; B) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{J(\Omega, A + tB) - J(\Omega, A)}{t}.$$

Again for the fixed space $X = H_0^1(\Omega)$

$$v \mapsto G(t, v) \stackrel{\text{def}}{=} E(\Omega, A + tB; v) : X \to \mathbb{R}, \quad g(t) \stackrel{\text{def}}{=} \inf_{v \in X} G(t, v), \quad dJ(\Omega, A; B) = dg(0).$$

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The objective is twofold: firstly to revisit the assumptions used in [42, Thm. 2.1, p. 394] dating back to 2001 in order to avoid conditions of the type $\liminf$'s of t-derivatives of $G(t,x)$ and secondly to catch the extra term.

For instance, if the set of minimizers $X(0)=\{x^0\}$ at $t=0$ is a singleton, for $t>0$

$$\frac{g(t)-g(0)}{t} = \underbrace{\frac{g(t)-G(t,x^0)}{t}}_{\le 0} + \underbrace{\frac{G(t,x^0)-G(0,x^0)}{t}}_{\to d_tG(0,x^0)=\lim_{t\searrow 0}\frac{G(t,x^0)-G(0,x^0)}{t}}$$

$$dg(0)=\lim_{t\searrow 0}\frac{g(t)-g(0)}{t} = \underbrace{\lim_{t\searrow 0}\frac{g(t)-G(t,x^0)}{t}}_{=R(x^0)\le 0} + d_tG(0,x^0).$$

What makes the next two theorems very attractive is that there is a priori no assumption on the set X or the differentiability of $G(t,x)$ with respect to x. In particular, they can be used for continuous convex non-differentiable functions $x\mapsto G(t,x)$ as we shall see in a series of simple examples.

The new theorem and its two subsequent specialized versions, which respectively assume first and second order semi-differentiability of the function $x\mapsto G(t,x)$, set the stage to handle perturbations of the form ([28, 31, 32])

(i) perturbations $x+tv$ in a vector space,

(ii) admissible trajectories in a group of diffeomorphisms, and

(ii) admissible semitrajectories in the group $X(D)$ of characteristic functions.

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(i) perturbations $x + tv$ in a vector space,

(ii) admissible trajectories in a group of diffeomorphisms, and

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We first consider the case where the extra term is zero.

THEOREM (NO EXTRA TERM)

Let X be an arbitrary set, $\tau > 0$, $(t,x) \mapsto G(t,x) : [0,\tau[\times X \to \mathbb{R}$, and

$$g(t) \stackrel{\text{def}}{=} \inf_{x\in X} G(t,x), \quad X(t) \stackrel{\text{def}}{=} \{x \in X : G(t,x) = g(t)\}, \quad 0 \le t < \tau.$$

Assume that the following conditions are satisfied:

(H1) *for all $t \in [0,\tau[$, $X(t) \neq \varnothing$;*

(H2) *for each $x^0 \in X(0)$ the one-sided t-derivative*

$$d_t G(0,x^0) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{G(t,x^0) - G(0,x^0)}{t} \text{ exists and is finite;} \tag{4.18}$$

(H3) *for each $t_n \searrow 0$, there exists $x^0 \in X(0)$ such that*

$$\lim_{n\to\infty} \frac{g(t_n) - G(t_n,x^0)}{t_n} = 0. \tag{4.19}$$

Then there exists $\bar{x}^0 \in X(0)$ such that[a]

$$dg(0) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{g(t) - g(0)}{t} = \inf_{x^0\in X(0)} d_t G(0,x^0) = d_t G(0,\bar{x}^0). \tag{4.20}$$

If, in addition, $X(0)$ is a singleton, the infimum can be dropped.

Remark

Theorem 19 is a generalization of [43, Thm. 2.1, p. 524] first formulated in [42, Thm. 2.1, p. 394] in 2001. It was recently used for eigenvalue problems (see [16] for elasticity theory in 2021 and [17] for the case of Steklov or Wentzell boundary conditions in 2022).

It relaxes the stronger assumptions ($\bar{\text{H}}$2), ($\bar{\text{H}}$3), and ($\bar{\text{H}}$4) that we briefly recall:

($\bar{\text{H}}$2) for all x in $\bigcup_{s\in[0,\tau[} X(s)$ and $t \in [0,\tau)$, the *t-derivative*

$$d_t G(t,x) \overset{\text{def}}{=} \lim_{\theta\to 0,\ 0\le t+\theta<\tau} \frac{G(t+\theta,x)-G(t,x)}{\theta} \text{ exists;} \tag{4.21}$$

($\bar{\text{H}}$3) for each $t_n \searrow 0$, there exist $x^0 \in X(0)$ and $\{x_n\}$, $x_n \in X(t_n)$, such that

$$\liminf_{n\to\infty\ s\searrow 0} d_t G(s,x_n) \ge d_t G(0,x^0); \tag{4.22}$$

($\bar{\text{H}}$4) for all x in $X(0)$, the map $t \mapsto d_t G(t,x)$ is upper semicontinuous at $t=0$.

Assumption ($\bar{\text{H}}$4) turns out to be unnecessary and assumptions ($\bar{\text{H}}$2) and ($\bar{\text{H}}$3) are replaced by the weaker and simpler assumptions (H2) and (H3) of Theorem 19, which only require the existence of $d_t G(0,x^0)$ at all $x^0 \in X(0)$. Condition (4.19) in Assumption (H3) only involving $G(t,x)$ is easier to check than the condition in the older Assumption ($\bar{\text{H}}$3) involving the t-derivative of $G(t,x)$.

Theorem (General case: occurenece of the extra term $R(x^0)$)

Let X be an arbitrary set, $\tau > 0$, $(t,x) \mapsto G(t,x) : [0,\tau[\times X \to \mathbb{R}$, and

$$g(t) \overset{\text{def}}{=} \inf_{x\in X} G(t,x), \quad X(t) \overset{\text{def}}{=} \{x \in X : G(t,x) = g(t)\}, \quad 0 \le t < \tau.$$

Assume that the following conditions are satisfied:

(H1) *for all $t \in [0,\tau[$, $g(t)$ is finite and $X(t) \neq \varnothing$;*

(H2) *for each $x^0 \in X(0)$, the one-sided t-derivative of $G(t,x^0)$ at $t = 0$,*

$$d_t G(0,x^0) \overset{\text{def}}{=} \lim_{\theta\searrow 0} \frac{G(\theta,x^0) - G(0,x^0)}{\theta} \quad \textit{exists and is finite;} \tag{4.23}$$

(H3) *for each $t_n \searrow 0$, there exists $x^0 \in X(0)$ such that*[a]

$$\lim_{n\to\infty} \frac{g(t_n) - G(t_n,x^0)}{t_n} = R(x^0), \tag{4.24}$$

$$\textit{where } x^0 \mapsto R(x^0) \overset{\text{def}}{=} \limsup_{t\searrow 0} \frac{g(t) - G(t,x^0)}{t} : X(0) \to [-\infty, 0]. \tag{4.25}$$

Then, $dg(0)$ exists and there exists $\bar{x}^0 \in X(0)$ such that[b]

$$dg(0) = \inf_{x^0\in X(0)} \left[d_t G(0,x^0) + R(x^0) \right] = d_t G(0,\bar{x}^0) + R(\bar{x}^0). \tag{4.26}$$

If, in addition, $X(0)$ is a singleton, the infimum can be dropped.

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Perturb the bounded open domain Ω by a family of diffeomorphisms T_t generated by a smooth velocity field $V(t)$:

$$\boxed{\Omega_t \overset{\text{def}}{=} T_t(\Omega)}, \quad T_t(X) \overset{\text{def}}{=} x(t;X),\ t \geq 0, \quad \frac{dx}{dt}(t;X) = V(t, x(t;X)),\ x(0;X) = X.$$

Given $f \in H^1(\mathbb{R}^n)$, consider the volume integral and the change of variable T_t

$$J(\Omega_t) = \int_{\Omega_t} f\,dx = \int_\Omega f \circ T_t\, j_t\, dx. \quad j_t = \det DT_t, \qquad DT_t \text{ is the Jacobian matrix,}$$

$$dJ(\Omega; V) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t} = \int_\Omega \nabla f \circ V(0) + f \operatorname{div} V(0)\, dx = \int_\Omega \operatorname{div}(f\, V(0))\, dx.$$

T_t will also be used in integrals involving functions u_t and v_t in $H^1(\Omega_t)$ to obtain an integral over Ω and functions $u^t = u_t \circ T_t$ and $v^t = v_t \circ T_t$ in the fixed space $H^1(\Omega)$:

$$\int_{\Omega_t} \nabla u_t \cdot \nabla v_t - a\, v\, dx = \int_\Omega \left[A(t) \nabla u^t \cdot \nabla v^t - a \circ T_t\, v^t j_t \right] dx \tag{4.27}$$

$$A(t) = j_t\, DT_t^{-1} (DT_t^{-1})^\top, \quad j_t = \det DT_t, \quad DT_t \text{ is the Jacobian matrix,} \tag{4.28}$$

where $(DT_t^{-1})^\top$ is the transpose of the inverse of DT_t.

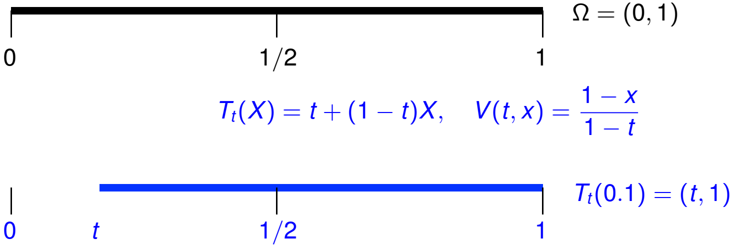

FIGURE 6: Domains $\Omega = (0, 1)$ and $\Omega_t = T_t(\Omega) = (t, 1)$

$$DT_t = j_t = 1 - t, \quad A(t) = j_t [DT_t]^{-1} [DT_t]^{-\top} = \frac{1}{1-t}, \quad A'(t) = \frac{1}{(1-t)^2}, \tag{4.29}$$

$$\text{div}\, V(t) = -\frac{1}{1-t}, \quad \text{div}\, V(0) = -\left. \frac{1}{1-t} \right|_{t=0} = -1, \quad A'(0) = \left. \frac{1}{(1-t)^2} \right|_{t=0} = 1$$

Example (Example 1 with an Extra Term)

Let $\Omega = (0,1) \subset \mathbb{R}$ and $u^0 \in V(\Omega) = \{v \in H^1(0,1) : v(1) = 2/3\}$ be the minimizing element for the compliance

$$\inf_{v \in V(\Omega)} E(\Omega, v), \quad E(\Omega, v) \stackrel{\text{def}}{=} \int_0^1 \left[\frac{1}{2}|v'|^2 - f\,v\right] dx, \quad f(x) \stackrel{\text{def}}{=} -\frac{1}{2}\frac{1}{x^{1/2}},$$

where f belongs to $L^{2-\varepsilon}(0,1)$, $0 < \varepsilon < 1$, but not to $L^2(0,1)$. Then u^0 is a solution of

$$(u^0)'' + f = 0 \text{ in } (0,1), \quad u^0(1) = 2/3 \text{ and } (u^0)'(0) = 0, \tag{4.30}$$

$$\Rightarrow \boxed{u^0(x) = \frac{2}{3}x^{3/2},} \quad (u^0)'(x) = x^{1/2}, \quad (u^0)''(x) = \frac{1}{2}\frac{1}{x^{1/2}}. \tag{4.31}$$

For $0 \le t < 1$, choose the transformation T_t and the velocity field $V(t,x)$

$$X \mapsto T_t(X) = t + (1-t)X : [0,1] \to [t,1], \quad V(t,x) = \frac{1-x}{1-t}. \tag{4.32}$$

Then $\Omega_t = T_t((0,1)) = (t,1)$, $V(\Omega_t) = \{v \in H^1(t,1) : v(1) = 2/3\}$, and

$$\inf_{v \in V(\Omega_t)} E(\Omega_t, v), \quad E(\Omega_t, v) \stackrel{\text{def}}{=} \int_t^1 \left[\frac{1}{2}|v'|^2 - f\,v\right] dx,$$

Example (Example 1 with an Extra Term)

The minimizer $u_t \in V(\Omega_t)$ is the solution of

$$(u_t)'' + f = 0 \text{ in } (t,1), \quad u_t(1) = 2/3,\ u_t'(t) = 0, \tag{4.33}$$

$$\Rightarrow \boxed{u_t(x) = t^{1/2}[1-x] + \frac{2}{3}x^{3/2},} \quad u_t'(x) = -t^{1/2} + x^{1/2}.$$

The function $g(t)$ can now be computed directly

$$\begin{aligned} g(t) &= \int_t^1 \left[\frac{1}{2}|u_t'|^2 - f\,u_t\right] dx \\ &= \int_t^1 \left[\frac{1}{2}|-t^{1/2} + x^{1/2}|^2 + \frac{1}{2}\frac{1}{x^{1/2}}\left(t^{1/2}[1-x] + \frac{2}{3}x^{3/2}\right)\right] dx \end{aligned}$$

$$g(t) = -\frac{t}{2} + \frac{5}{12} - \frac{t^2}{12} \quad \Rightarrow g'(t) = -\frac{1}{2} - \frac{t}{6}, \text{ and } \boxed{dg(0) = -\frac{1}{2}.} \tag{4.34}$$

Example (Example 1 with an Extra Term)

For the (right-hand side) t-derivative $d_t G(0, u_0)$ at $t = 0$ of

$$G(t, u_0) = E\left(T_t(\Omega), u_0 \circ T_t^{-1}\right) = \int_\Omega \frac{1}{2} A(t)\nabla u_0 \cdot \nabla u_0 - j_t (f \circ T_t) u_0 \, dx, \tag{4.35}$$

$$d_t G(0, u_0) = \int_\Omega \left[\frac{1}{2} A'(0)\nabla u_0 \cdot \nabla u_0 - \left[(\operatorname{div} V(0))\, f + \nabla f \cdot V(0)\right] u_0\right] dx. \tag{4.36}$$

where for the transformation T_t and velocity field $V(t, x)$ chosen in (4.32)

$$DT_t = j_t = 1 - t, \quad A(t) = j_t [DT_t]^{-1} [DT_t]^{-\top} = \frac{1}{1-t}, \quad A'(t) = \frac{1}{(1-t)^2}, \tag{4.37}$$

$$\operatorname{div} V(t) = -\frac{1}{1-t}, \quad \operatorname{div} V(0) = -\left.\frac{1}{1-t}\right|_{t=0} = -1, \quad A'(0) = \left.\frac{1}{(1-t)^2}\right|_{t=0} = 1$$

$$\begin{aligned} d_t G(0, u_0) &= \int_\Omega \left[\frac{1}{2}x - \left(\frac{1}{2x^{1/2}} + \frac{1}{4}\frac{1}{x^{3/2}}(1-x)\right)\frac{2}{3}x^{3/2}\right] dx \\ &= \int_0^1 \left[\frac{1}{2}x - \frac{2}{3}\left(\frac{x}{2} + \frac{1}{4}(1-x)\right)\right] dx = \int_0^1 \left[\frac{1}{3}x - \frac{1}{6}\right] dx = 0. \end{aligned} \tag{4.38}$$

So we don't recover the previously computed $dg(0) = -1/2$.
There is a missing negative term as in the example provided in Delfour-Sturm [38, sec. 2.5, pp. 145–148] of a constrained objective function.

Example (Example 1 with an Extra Term in Higher Dimensions)

This example can be extended to higher dimensions. For instance In $\mathbb{R}^2$, let

$$\boxed{\Omega = (0,1)\times(0,1), \quad \Gamma_1 = \{(1,y) : 0 \le y \le 1\}.}$$

Let $u^0 \in V(\Omega) = \{v \in H^1(\Omega) : v = 2/3 \text{ on } \Gamma_1\}$ be the unique minimizing element for the compliance

$$\inf_{v \in V(\Omega)} E(\Omega, v), \quad E(\Omega, v) \overset{\text{def}}{=} \int_\Omega \left[\frac{1}{2}\|\nabla v\|^2 - f\,v\right] dx, \quad f(x,y) \overset{\text{def}}{=} -\frac{1}{2}\frac{1}{x^{1/2}}.$$

The minimizer u_0 is solution of the problem

$$\Delta u_0(x,y) - \frac{1}{2}\frac{1}{x^{1/2}} = 0 \text{ in } \Omega, \quad u_0|_{\Gamma_1} = 2/3, \quad \frac{\partial u_0}{\partial n} = 0 \text{ on } \Gamma\backslash\Gamma_1, \tag{4.39}$$

and the minimizer is $u_0(x,y) = (2/3)\,x^{3/2}$. For $0 \le t < 1$, choose the transformation

$$(X,Y) \mapsto T_t(X,Y) = (t + (1-t)X, Y) : [0,1]\times[0,1] \to [t,1]\times[0,1] \tag{4.40}$$

$$\Omega_t \overset{\text{def}}{=} T_t(\Omega) = \{(x,y) : x \in (t,1),\ y \in (0,1)\}. \tag{4.41}$$

The next example involves a convex continuous non-differentiable functional.

Example (Example 2. The extra term is zero)

Let $a > 0$, $b \in \mathbb{R}$, $\Omega = (0,1)$, and for $v \in H_0^1(0,1)$ the convex continuous non-differentiable function

$$\inf_{v\in H_0^1(0,1)} E(\Omega, v), \quad E(\Omega, v) \stackrel{\text{def}}{=} \int_0^1 \left[(|v'| - a)^2 + b \right] dx. \tag{4.42}$$

The function

$$u_0(x) = a\left(\left| x - \frac{1}{2} \right| - \frac{1}{2} \right) \tag{4.43}$$

is a minimizer, but it is not unique. The minimizers are characterized by

$$|u_0'(x)| = a \quad \text{a.e in } (0,1), \quad u_0(0) = u_0(1) = 0 \tag{4.44}$$

$$\Rightarrow g(0) = \inf_{v\in H_0^1(0,1)} E(\Omega, v) = \int_0^1 (|u_0'| - a)^2 + b\, dx = b. \tag{4.45}$$

For the perturbed problem indexed by $0 \le t < 1$ choose the shape perturbation

$$T_t(X) = t + (1-t)X, \quad V(t,x) = \frac{1-x}{1-t}, \quad DT_t = 1 - t. \tag{4.46}$$

Therefore, $T_t(0.1) = (t, 1)$.

EXAMPLE (EXAMPLE 2. THE EXTRA TERM IS ZERO)

Therefore, the minimization problem on $T_t(0.1) = (t, 1)$ is

$$\inf_{v\in H_0^1(t,1)} E(\Omega_t, v), \quad E(\Omega_t, v) \stackrel{\text{def}}{=} \int_t^1 \left(|v'| - a\right)^2 + b\, dx. \tag{4.47}$$

The function

$$u_t(x) = a\left(\left|x - \frac{1+t}{2}\right| - \frac{1-t}{2}\right) \tag{4.48}$$

is a minimizer, but it is not unique. The minimizers are characterized by

$$|u_t'(x)| = a \quad \text{a.e in } (t, 1), \quad u_0(t) = u_0(1) = 0 \tag{4.49}$$

$$\Rightarrow g(t) = \inf_{v\in H_0^1(t,1)} E(\Omega_t, v) = \int_t^1 \left(|u_t'| - a\right)^2 + b\, dx = (1-t)b, \tag{4.50}$$

and $dg(0) = -b$. For the t-derivative and $u_0 \in H_0^1(0, 1)$

$$\begin{aligned} G(t, u_0) &= E\left(T_t(\Omega), u_0 \circ T_t^{-1}\right) = \int_0^1 \left[\left(\frac{1}{1-t}|u_0'| - a\right)^2 + b\right](1-t)\, dx \\ &= \int_0^1 \left[\left(\frac{1}{1-t}a - a\right)^2 + b\right](1-t)\, dx = a^2\frac{t^2}{1-t} + b\,(1-t) \end{aligned} \tag{4.51}$$

and $d_tG(0, u_0) = -b$.

EXAMPLE (EXAMPLE 2. THE EXTRA TERM IS ZERO)

Finally, Hypothesis (H3) of Theorem 19 is verified:

$$\frac{g(t)-G(t,u_0)}{t} = \frac{(1-t)b - \left[a^2\frac{t^2}{1-t} + b(1-t)\right]}{t} = -a^2\frac{t}{1-t}$$

$$\Rightarrow R(u_0) = \lim_{t\searrow 0}\frac{g(t)-G(t,u_0)}{t} = \lim_{t\searrow 0} -a^2\frac{t}{1-t} = 0.$$

Since the expressions of $R(u_0)=0$ and $d_tG(0,u_0)=-b$ are independent of the choice of $u_0\in X(0)$, the infimum in the expression of $dg(0)$ can be dropped even if $X(0)$ is not a singleton.

Example (Example 4 with an Extra Term)

Consider a variant of Example 25 for a slightly different convex continuous non-differentiable function $v \mapsto E(\Omega, v)$ to be minimized over $H_0^1(0,1)$

$$\inf_{v \in H_0^1(0,1)} E(\Omega, v), \quad E(\Omega, v) \stackrel{\text{def}}{=} \int_0^1 \big||v'| - a\big| + b\, dx, \quad a > 0, \quad b \in \mathbb{R}\,. \tag{4.52}$$

The minimizers are the same as in Example 25 and $g(0) = b$. For the same perturbations (4.46) the minimization problem parametrized by $0 \leq t < 1$ is

$$\inf_{v \in H_0^1(t1)} E(\Omega_t, v), \quad E(\Omega_t, v) \stackrel{\text{def}}{=} \int_t^1 \big||v'| - a\big| + b\, dx. \tag{4.53}$$

It has the same minimizers as in Example 25, $g(t) = (1-t)b$, and $dg(0) = -b$. For the t-derivative with $u_0 \in H_0^1(0,1)$

$$\begin{aligned} G(t, u_0) \stackrel{\text{def}}{=} E(\Omega_t, u_0 \circ T_t^{-1}) &= \int_0^1 \left(\left| \frac{1}{1-t} |u_0'| - a \right| + b \right) (1-t)\, dx \\ &= \int_0^1 \left(\left| \frac{1}{1-t} a - a \right| + b \right) (1-t)\, dx = a\frac{t}{1-t} + b(1-t) \end{aligned}$$

and $d_t G(0, u_0) = a - b$.

Example (Example 4 with an Extra Term)

Hypothesis (H3) of Theorem 20 is satisfied,

$$\frac{g(t) - G(t,u_0)}{t} = \frac{(1-t)b - \left[a\frac{t}{1-t} + b(1-t)\right]}{t} = -a\frac{1}{1-t}$$

$$\Rightarrow R(u_0) = \lim_{t \searrow 0} \frac{g(t) - G(t,u_0)}{t} = \lim_{t \searrow 0} -a\frac{1}{1-t} = -a < 0,$$

and there is a non-zero extra term $R(u_0) = -a$.
The terms $R(u_0) = -a$ and $d_t G(0,u_0) = a - b$ are independent of the choice of $u_0 \in X(0)$.
Therefore the infimum in the expression of $dg(0) = -b$ can be dropped even if $X(0)$ is not a singleton.

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The minimizers can often be characterized by a variational equation or inequality when the function $x \mapsto G(t, x)$ enjoys some form of differentiability.

For instance, If $x \in A$ is a local minimizer of f with respect to A, then

$$\boxed{d_H f(x; v) \geq 0 \text{ for all } v \in T_A^\flat(x), \quad \text{variational inequality}} \tag{5.1}$$

When $T_A^\flat(x)$ is linear and $v \mapsto d_H f(x; v)$ is linear, then

$$\boxed{d_H f(x; v) = 0 \text{ for all } v \in T_A^\flat(x), \quad \text{variational equation.}} \tag{5.2}$$

This differentiability assumption can be used to characterize the negative extra term.

To keep things simple we assume that X is respectively convex and affine.

Let X be a convex subset of a vector space. For $t>0$, $x^0 \in X(0)$, and $x^t \in X(t)$, let $\theta \mapsto G(t, x^0 + \theta(x^t - x^0))$ on $[0,1]$ satisfy

$$g(t) = G(t,x^t) = G(t,x^0) + \int_0^1 d_x G(t, x+\theta(x^t-x^0); x^t - x^0)\, d\theta.$$

As a consequence, for $t>0$, $x^0 \in X(0)$, and $x^t \in X(t)$

$$\frac{g(t)-g(0)}{t} = \int_0^1 d_x G\left(t, x^0+\theta(x^t-x^0); \frac{x^t-x^0}{t}\right) d\theta + \frac{G(t,x^0)-G(0,x^0)}{t},$$

where the second term would converge to $d_t G(0; x^0)$ as $t>0$ goes to zero.

HYPOTHESIS (H0)

Let X be a convex subset of a vector space, $(t,x) \mapsto G(t,x) : [0,\tau[\times X \to \mathbb{R}$, $\tau > 0$, and the associated minimization problems

$$g(t) \overset{\text{def}}{=} \inf_{x \in X} G(t,x), \quad X(t) \overset{\text{def}}{=} \{x \in X : G(t,x) = g(t)\}, \quad 0 \le t < \tau.$$

Assume that

(i) for all $t \in [0,\tau[$ and $x, y \in X$ the following limit exists

$$d_x G(t, x; y-x) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{G(t, x+\theta(y-x)) - G(t,x)}{\theta} \tag{5.3}$$

(ii) and, for all $t \in [0,\tau[$, $x^0 \in X(0)$, and $x^t \in X(t)$, the function

$$\theta \mapsto G(t, x^0+\theta(x^t-x^0)) : [0,1] \to \mathbb{R} \quad \text{is absolutely continuous.} \tag{5.4}$$

Let X be a convex subset of a vector space. For $t>0$, $x^0 \in X(0)$, and $x^t \in X(t)$, let $\theta \mapsto G(t, x^0 + \theta(x^t - x^0))$ on $[0,1]$ satisfy

$$g(t) = G(t,x^t) = G(t,x^0) + \int_0^1 d_x G(t, x + \theta(x^t - x^0); x^t - x^0)\, d\theta.$$

As a consequence, for $t>0$, $x^0 \in X(0)$, and $x^t \in X(t)$

$$\frac{g(t)-g(0)}{t} = \int_0^1 d_x G\left(t, x^0 + \theta(x^t - x^0); \frac{x^t - x^0}{t}\right) d\theta + \frac{G(t,x^0) - G(0,x^0)}{t},$$

where the second term would converge to $d_t G(0;x^0)$ as $t>0$ goes to zero.

Hypothesis (H0)

Let X be a convex subset of a vector space, $(t,x) \mapsto G(t,x) : [0,\tau[\times X \to \mathbb{R}$, $\tau > 0$, and the associated minimization problems

$$g(t) \overset{\text{def}}{=} \inf_{x \in X} G(t,x), \quad X(t) \overset{\text{def}}{=} \{x \in X : G(t,x) = g(t)\}, \quad 0 \le t < \tau.$$

Assume that

(i) for all $t \in [0,\tau[$ and $x, y \in X$ the following limit exists

$$d_x G(t,x;y-x) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{G(t, x + \theta(y-x)) - G(t,x)}{\theta} \tag{5.3}$$

(ii) and, for all $t \in [0,\tau[$, $x^0 \in X(0)$, and $x^t \in X(t)$, the function

$$\theta \mapsto G(t, x^0 + \theta(x^t - x^0)) : [0,1] \to \mathbb{R} \quad \text{is absolutely continuous.} \tag{5.4}$$

THEOREM (X BE A CONVEX SUBSET OF A VECTOR SPACE)

Let (H0) and the following hypotheses be satisfied:

(H1) *for all* $t \in [0,\tau[$, $g(t)$ *is finite,* $X(t) \neq \varnothing$;

(H2) *for each* $x^0 \in X(0)$, *the one-sided* t*-derivative of* $G(t,x^0)$ *at* $t = 0$,

$$d_t G(0,x^0) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{G(\theta,x^0) - G(0,x^0)}{\theta} \quad \text{exists and is finite;} \tag{5.5}$$

(H3) *for each* $t_n \searrow 0$, *there exist* $x^0 \in X(0)$ *and* $\{x_n\}$, $x_n \in X(t_n)$, *such that*

$$\lim_{n\to\infty} \int_0^1 d_x G\left(t_n, x^0 + \theta\left(x_n - x^0\right); \frac{x_n - x^0}{t_n}\right) d\theta = R(x^0),$$

where $R : X(0) \to [-\infty, 0]$ *is defined as*

$$R(x^0) \overset{\text{def}}{=} \limsup_{t \searrow 0} \frac{g(t) - G(t,x^0)}{t}. \tag{5.6}$$

Then, $dg(0)$ *exists and there exists* $\bar{x}^0 \in X(0)$ *such that*

$$dg(0) = \inf_{x^0 \in X(0)} \left[d_t G(0,x^0) + R(x^0)\right] = d_t G(0,\bar{x}^0) + R(\bar{x}^0). \tag{5.7}$$

If, in addition, $X(0)$ *is a singleton, the infimum can be dropped.*

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Let X be an affine subspace of a vector space $\mathcal{X}$, S the unique linear subspace associated with X. For $t > 0$, $x^0 \in X(0)$, and $x^t \in X(t)$, $x^0 - x^t \in S$ and

$$G(t, x^0) = G(t, x^t) + \underbrace{d_x G\left(t, x^t; x^0 - x^t\right)}_{=0}$$

$$+ \frac{1}{2} \int_0^1 d_x^2 G\left(t, x^t + \theta\left(x^0 - x^t\right); x^0 - x^t; x^0 - x^t\right) d\theta$$

since, for a minimizer $x^t \in X(t)$ at t,

$$\forall v \in S, \quad d_x G(t, x^t; v) = 0$$

$$\Rightarrow \frac{g(t) - G(t, x^0)}{t} =$$

$$\frac{G(t, x^t) - G(t, x^0)}{t} = -\frac{1}{2} \int_0^1 d_x^2 G\left(t, x^t + \theta\left(x^0 - x^t\right); \frac{x^0 - x^t}{t^{1/2}}; \frac{x^0 - x^t}{t^{1/2}}\right) d\theta.$$

$$\frac{g(t) - g(0)}{t}$$

$$= -\frac{1}{2} \int_0^1 d_x^2 G\left(t, x^t + \theta\left(x^0 - x^t\right); \frac{x^0 - x^t}{t^{1/2}}; \frac{x^0 - x^t}{t^{1/2}}\right) d\theta + \frac{G(t, x^0) - G(0, x^0)}{t},$$

where the second term would converge to $d_t G(0; x^0)$ as $t > 0$ goes to zero.

Hypothesis (A0)

Let X be an affine subspace of a vector space $\mathcal{X}$, S the unique linear subspace associated with X, $(t,x) \mapsto G(t,x) : [0,\tau[\times X \to \mathbb{R}$ for some $\tau > 0$. Assume that

(i) for all t, $x \in X$, and $v, w \in S$ the following semidifferentials

$$d_x G(t,x;v) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{G(t,x+\theta v) - G(t,x)}{\theta}$$

$$d_x^2 G(t,x;v;w) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{d_x G(t,x+\theta w;v) - d_x G(t,x;v)}{\theta}$$

exist, that the function $v \mapsto d_x G(t,x;v) : S \to \mathbb{R}$ is linear,

(ii) and, for all $t \in [0,\tau[$, $x^0 \in X(0)$, and $x^t \in X(t)$, the function $\theta \mapsto G(t, x^t + \theta(x^0 - x^t)) : [0,1] \to \mathbb{R}$ and its derivative are absolutely continuous.

THEOREM (X BE AN AFFINE SUBSPACE OF A VECTOR SPACE)

Let (A0) and the following hypotheses be satisfied:

(H1) *for all* $t \in [0, \tau[$, $g(t)$ *is finite,* $X(t) \neq \varnothing$;

(H2) *for each* $x^0 \in X(0)$, *the one-sided* t*-derivative of* $G(t, x^0)$ *at* $t = 0$,

$$d_t G(0, x^0) \stackrel{\text{def}}{=} \lim_{\theta \searrow 0} \frac{G(\theta, x^0) - G(0, x^0)}{\theta} \quad \text{exists and is finite;} \tag{5.8}$$

(H3) *for each* $t_n \searrow 0$, *there exist* $x^0 \in X(0)$ *and* $\{x_n\}$, $x_n \in X(t_n)$, *such that*

$$\lim_{n\to\infty} -\frac{1}{2} \int_0^1 d_x^2 G\left(t_n, x_n + \theta\left(x^0 - x_n\right); \frac{x^0 - x_n}{t_n{}^{1/2}}; \frac{x^0 - x_n}{t_n{}^{1/2}}\right) d\theta = R(x^0),$$

where $R : X(0) \to [-\infty, 0]$ *is defined as*

$$R(x^0) \stackrel{\text{def}}{=} \limsup_{t \searrow 0} \frac{g(t) - G(t, x^0)}{t}. \tag{5.9}$$

Then, $dg(0)$ *exists and there exists* $\bar{x}^0 \in X(0)$ *such that*

$$dg(0) = \inf_{x^0 \in X(0)} \left[d_t G(0, x^0) + R(x^0)\right] = d_t G(0, \bar{x}^0) + R(\bar{x}^0). \tag{5.10}$$

If, in addition, $X(0)$ *is a singleton, the infimum can be dropped.*

Example (Shape derivative of the earlier Example 21)

We now go back to Example 21 of a shape derivative. Let $\Omega = (0,1) \subset \mathbb{R}$, $\Gamma = \{0,1\}$, and $u^0 \in V(\Omega) = \{v \in H^1(0,1) : v(1) = 2/3\}$ be the minimizer

$$\inf_{v \in V(\Omega)} E(\Omega, v), \quad E(\Omega, v) \overset{\text{def}}{=} \int_0^1 \left[\frac{1}{2}|v'|^2 - f\,v\right] dx, \quad f(x) \overset{\text{def}}{=} -\frac{1}{2}\frac{1}{x^{1/2}},$$

and use Theorem 31 to compute the missing extra term:

$$d_x^2 G(t,u;v;v) = \int_\Omega A(t)\nabla v \cdot \nabla v\, dx = \int_0^1 A(t)\nabla v \cdot \nabla v\, dx$$

$$-R(u^0) = \lim_{t \searrow 0} \frac{1}{2}\int_0^1 A(t)\frac{(u^t - u^0)'}{t^{1/2}}\,\frac{(u^t - u^0)'}{t^{1/2}}\, dx = \lim_{t \searrow 0}(1-t)\frac{1}{2}\int_0^1 \left|\frac{(u^t - u^0)'}{t^{1/2}}\right|^2 dx.$$

The derivative of $u^t = u_t \circ T_t$ is $(u^t)'(x) = DT_t(x)(u_t)'(T_t(x))$

$$(u^t)'(x) = (1-t)\left[(t + (1-t)x)^{1/2} - t^{1/2}\right]$$

$$\Rightarrow \frac{(u^t - u^0)'}{t^{1/2}}(x) \to -1 \text{ in } L^2(0,1)\text{-norm}, \quad \frac{u^t - u^0}{t^{1/2}}(x) \to 1 - x \text{ in } H^1(0,1)\text{-norm}.$$

$R(u^0) = -1/2$ corrects the t-derivative $d_t G(0,u^0) = 0$ to give $dg(0) = -1/2$ as predicted by the previous direct computation (4.34) of $dg(0)$.

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Let Ω be a bounded open in $\mathbb{R}^n$ with Lipschitz boundary Γ, $b \in H^{1/2}(\Gamma)$, $f \in L^2(\Omega)$,

$$V_b(\Omega) \stackrel{\text{def}}{=} \{v \in H^1(\Omega) : v|_\Gamma = b\} \quad \Rightarrow \quad V_0(\Omega) = H_0^1(\Omega). \tag{6.1}$$

Let u_0 be the minimizer

$$F(\Omega) = \inf_{v \in V_b(\Omega)} F(\Omega; v), \quad F(\Omega; v) \stackrel{\text{def}}{=} \int_\Omega \frac{1}{2}\|\nabla v\|^2 - f\, v\, dx, \tag{6.2}$$

$$u_0 \in V_b(\Omega), \forall v \in V_0(\Omega, \quad \int_\Omega \nabla u_0 \cdot \nabla v - fv\, dx = 0,$$

$$\Rightarrow \boxed{u_0 \in V_b(\Omega), \quad \Delta u_0 + f = 0, \quad u_0 = b \text{ on } \Gamma.} \tag{6.3}$$

$E = \{\bar{x}\}$

$E_r = \overline{B_r(\bar{x})} \subset \Omega$

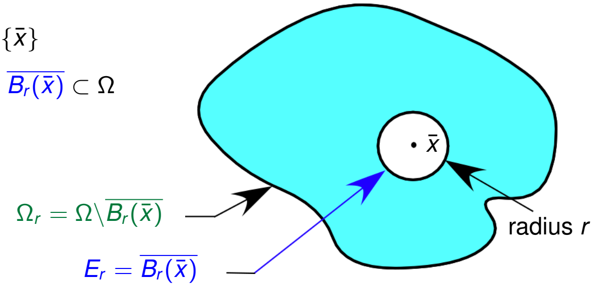


FIGURE 7: Dilated set E_r and perturbed domain $\Omega_r = \Omega \backslash E_r$

Associate with r, $0 < r \leq R$, the *perturbed domain* $\Omega_r = \Omega \backslash E_r$, where, by assumption, $\partial\Omega_r = \Gamma \cup \partial E_r$, $\Gamma \cap \partial E_r = \varnothing$, and ∂E_r is $C^{1,1}$.

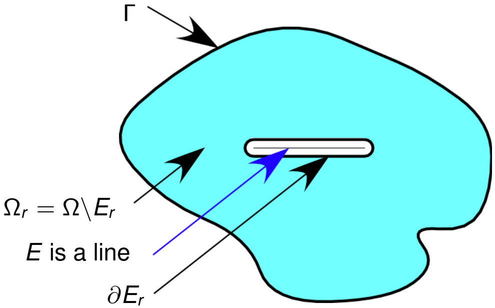


FIGURE 8: Dilated set E_r and perturbed domain Ω_r with a single connected component

Let $u_r \in V_b(\Omega_r) = \{v \in H^1(\Omega_r) : v = b \text{ on } \Gamma\}$ be the solution of the problem

$$F(\Omega_r) = \inf_{v \in V_b(\Omega_r)} F(\Omega_r; v), \quad F(\Omega_r; v) \overset{\text{def}}{=} \int_{\Omega_r} \frac{1}{2}\|\nabla v\|^2 - f\, v\, dx, \tag{6.4}$$

$$\exists u_r \in V_b(\Omega_r), \forall v \in V_0(\Omega_r), \quad \int_{\Omega_r} \nabla u_r \cdot \nabla v - f\, v\, dx = 0, \tag{6.5}$$

$$\Rightarrow \boxed{-\Delta u_r = f \text{ in } \Omega_r, \quad u_r = b \text{ on } \Gamma \quad \text{and} \quad \frac{\partial u_r}{\partial n_{\Omega_r}} = 0 \text{ on } \partial E_r.} \tag{6.6}$$

The application of Theorem 31 of section 5.2 requires a fixed affine subset X of the space $H^1(\Omega)$. So, the solution $u_r \in V_b(\Omega_r) \subset H^1(\Omega_r)$ need to be extended to the affine subspace $V_b(\Omega) \subset H^1(\Omega)$.

Consider the extension $u^r : \Omega \to \mathbb{R}$ obtained by introducing the solution $u_r^o : E_r^o \to \mathbb{R}$ of the problem

$$\boxed{-\Delta u_r^o = f \text{ in } E_r^o, \quad u_r^o = u_r \text{ on } \partial E_r.} \tag{6.7}$$

By *near boundary regularity* near the boundary ∂E_r, for all $1 < p < \infty$, u_r belongs to $V_b(\Omega_r) \cap W^{2,p}(E_{2R}^o \backslash E_r)$ (cf. [61, Cor. 9.18, p. 243]). Therefore, $u_r \in V_b(\Omega_r) \cap C^{1,\alpha}(E_R \backslash E_r^o)$ for all $0 < \alpha < 1$.

The traces of u_r and ∇u_r belong to $C^{0,\alpha}(\partial E_r)$. But, since $\partial u_r / \partial n = 0$ on ∂E_r, the trace of ∇u_r coincides with the *tangential gradient* of u_r on ∂E_r and $u_r \in C^{1,\alpha}(\partial E_r)$. As a result, the problem (6.7) has a unique solution $u_r^o \in C^{1,\alpha}(E_r) \cap H^1(E_r^o)$. The extended fonction $u^r : \Omega \to \mathbb{R}$ constructed from u_r in Ω_r and u_r^o in E_r^o belongs to the *fixed space* $V_b(\Omega) \cap C^0(E_R)$ with a discontinuity in its normal derivative across ∂E_r.

We use Theorem 31 with $X = V_b(\Omega) \cap C^0(E_R)$ where the sets $X(t)$ for $t > 0$ are not singletons since the $H^1(\Omega)$-extension u^r is not unique for $r > 0$, but $X(0)$ is a singleton.

For the t-derivative of $G(t, u_0)$, since $u_0 \in V_b(\Omega) \cap C^1(E_R)$,

$$\frac{G(t,u_0) - G(0,u_0)}{t} = \frac{\int_{\Omega_r} \frac{1}{2}\|\nabla u_0\|^2 - f\, u_0\, dx - \int_{\Omega} \frac{1}{2}\|\nabla u_0\|^2 - f\, u_0\, dx}{t}$$

$$= -\frac{\int_{E_r^o} \frac{1}{2}\|\nabla u_0\|^2 - f\, u_0\, dx}{t}$$

$$\Rightarrow d_t G(0, x_0) = -\int_E \frac{1}{2}\|\nabla u_0\|^2 - f\, u_0\, dH^d.$$

For the extra term $R(u_0)$, we have

$$d_x^2 G(t, \varphi; \psi; \psi) = \int_{\Omega_r} \nabla\psi \cdot \nabla\psi\, dx$$

$$\Rightarrow \int_0^1 d_x^2 G\left(t, x^t + \theta(x^0 - x^t); \frac{x^0 - x^t}{t^{1/2}}; \frac{x^0 - x^t}{t^{1/2}}\right) d\theta$$

$$= \int_{\Omega_r} \nabla\left(\frac{u_r - u_0}{t^{1/2}}\right) \cdot \nabla\left(\frac{u_r - u_0}{t^{1/2}}\right) dx = \int_{\Omega_r} \left\|\nabla\left(\frac{u_r - u_0}{t^{1/2}}\right)\right\|^2 dx.$$

Thus, in view of Assumption (H3) of Theorem 31, the topological derivative exists if and only if the following limit exists:

$$R(u_0) = -\lim_{\substack{t=2r\\ r\searrow 0}} \frac{1}{2}\int_{\Omega_r} \left\|\nabla\left(\frac{u_r - u_0}{t^{1/2}}\right)\right\|^2 dx = \lim_{r\searrow 0} \frac{1}{2}\frac{1}{2r}\int_{\partial E_r} \frac{\partial u_0}{\partial n_{\Omega_r}}\,(u_r - u_0)\, dH^{n-1}.$$

ASSUMPTION

Let E be a closed connected subset of $\mathbb{R}^n$ and $R > 0$ such that
$E_{2R} = \{x \in \mathbb{R}^n : d_E(x) \leq 2R\} \subset \Omega$.
Assume that E has *positive reach* greater than $2R$ (reach $E > 2R$) and that
$0 < H^d(E) < \infty$ for an integer d, $0 \leq d < n$. Let $f \in L^2(\Omega) \cap C^0(E_{2R})$.

THEOREM ([29])

Let the above Assumption 6.1 be verified[a] *and* $t = \alpha_{n-d} r^{n-d}$.
The d-topological derivative with respect to the set E exists if and only if the following limit[b] *exists*

$$R(u_0) \stackrel{\text{def}}{=} -\lim_{t \searrow 0} \frac{1}{2} \int_{\Omega_r} \left\| \nabla \left(\frac{u_r - u_0}{t^{1/2}} \right) \right\|^2 dx \left(= \lim_{t \searrow 0} \frac{1}{2} \frac{1}{t} \int_{\partial E_r} \frac{\partial u_0}{\partial n_{\Omega_r}} (u_r - u_0)\, dH^{n-1} \right).$$

Then it is given by the formula

$$dF\left(\chi_\Omega; \delta_{E,H^d}\right) = d_t G(0, u_0) + R(u_0), \tag{6.8}$$

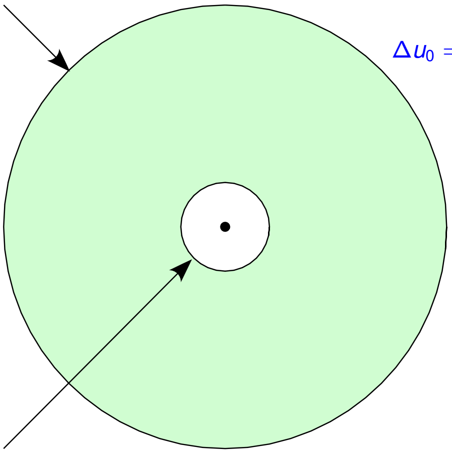
where

$$d_t G(0, u_0) = -\int_E \left(\frac{1}{2} \|\nabla u_0\|^2 - f\, u_0 \right) dH^d. \tag{6.9}$$

---

[a] Recall that $f \in L^2(\Omega) \cap C^0(E_{2R})$.

$\Omega = B_R(0) = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| < R\}$



$\Delta u_0 = 0$ in $B_R(0)$, $\quad u_0(x,y) = x$ on $\partial B_R(0)$

Perturbed domain $\quad \Omega_r = \Omega \backslash E_r$

$\Delta u_r = 0$ in Ω_r, $\quad u_r = x$ on $\partial B_R(0)$,

$\frac{\partial u_r}{\partial n} = 0$ on $\partial B_r(0)$

dilated set E_r

$E = \{(0,0)\}$ is a point

FIGURE 9: Domain Ω, dilated set E_r, and perturbed domain Ω_r

Examples in dimension 2.

EXAMPLE ([97, EXAMPLE 1, SEC. 3, P. 1258])

Let $\Omega = B_R(0) \subset \mathbb{R}^2$,

$$\Delta u_0 = 0 \text{ in } B_R(0), \quad u_0 = x \text{ on } \partial B_R(0) \quad \Rightarrow u_0(x,y) = x \text{ in } \overline{B_R(0)}. \tag{6.10}$$

For $E = \{0\}$ and $0 < r < R$, let $E_r = \overline{B_r(0)}$, $\Omega_r = B_R(0)\backslash E_r$, $u_r \in H^1(\Omega_r)$ solution of

$$\Delta u_r = 0 \text{ in } \Omega_r, \quad u_r = x \text{ on } \partial B_R(0), \quad \frac{\partial u_r}{\partial n} = 0 \text{ on } \partial B_r(0) \tag{6.11}$$

$$\Rightarrow u_r(x,y) = \frac{R^2}{R^2+r^2}\left(\frac{r^2}{x^2+y^2}+1\right) x \text{ in } \Omega_r \tag{6.12}$$

$$(u_r - u_0)(x,y) = \frac{R^2}{R^2+r^2}\left(\frac{r^2}{x^2+y^2}\right) x - \frac{r^2}{R^2+r^2}\, x = \frac{x\, r^2}{R^2+r^2}\left[\frac{R^2}{x^2+y^2} - 1\right]$$

$$\lim_{r\searrow 0} \int_{\Omega_r} \left|\frac{u_r - u_0}{t^{1/2}}\right|^2 dx = 0 \quad \text{and} \quad \lim_{r\searrow 0} \int_{\Omega_r} \left\|\nabla\left(\frac{u_r-u_0}{t^{1/2}}\right)\right\|^2 dx = 1 \tag{6.13}$$

$$\Rightarrow R(u_0) = -\frac{1}{2} \tag{6.14}$$

$$d_t G(0,u_0) = -\int_E \left(\frac{1}{2}\|\nabla u_0\|^2 - a\, u_0\right) dH^0 = -\frac{1}{2}, \qquad \boxed{dF(\chi_\Omega;\delta_E) = -\frac{1}{2} - \frac{1}{2} = -1.}$$

Example in dimension 2:

$$\Delta u_0 = 0 \text{ in } B_R(0), \quad u_0 = x \text{ on } \partial B_R(0)$$

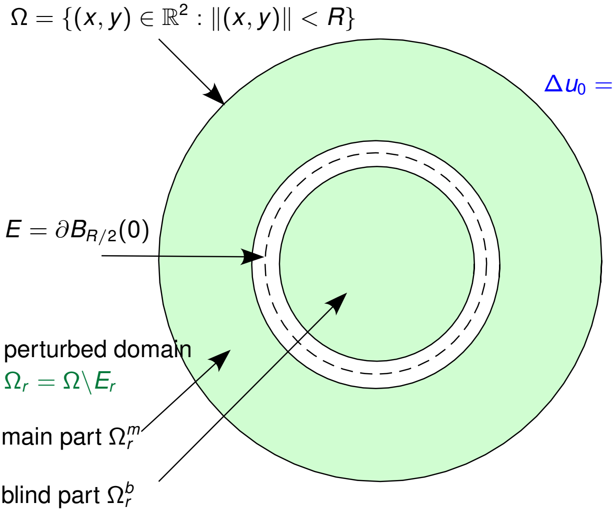


FIGURE 10: Domain Ω, dilated set E_r, and the two connected components Ω_r^m and Ω_r^b of the perturbed domain Ω_r

Example

Let $\Omega = B_R(0) \subset \mathbb{R}^2$, the open ball of radius R and

$$\Delta u_0 = 0 \text{ in } B_R(0), \quad u_0 = x \text{ on } \partial B_R(0) \quad \Rightarrow u_0(x,y) = x \text{ in } \overline{B_R(0)}. \tag{6.15}$$

Go back to Example 34 but with a circle $E = \partial B_{R/2}(0)$ of dimension $d = 1$ instead of a point $E = \{0\}$ of dimension $d = 0$.

For $0 < r < R/8$, the perturbed domain Ω_r has two connected components: $\Omega_r^m = B_R(0)\backslash\overline{B_{R/2+r}(0)}$ and $\Omega_r^b = B_{R/2-r}(0)$.

By choosing $u_r = u_0$ on Ω_r^b we only have to work with Ω_r^m. Let $u_r \in H^1(\Omega_r)$ be the solution of

$$\Delta u_r = 0 \text{ in } \Omega_r, \quad u_r = x \text{ on } \partial B_R(0), \quad \frac{\partial u_r}{\partial n} = 0 \text{ on } \partial B_r(0) \tag{6.16}$$

$$\Rightarrow u_r(x,y) = \begin{cases} \dfrac{R^2}{R^2+r^2}\left(\dfrac{r^2}{x^2+y^2}+1\right)x, & (x,y) \in B_R(0)\backslash\overline{B_{R/2+r}(0)} \\ u_0(x,y), & (x,y) \in B_{R/2-r}(0) \end{cases} \tag{6.17}$$

$$w_r(x,y) \stackrel{\text{def}}{=} (u_r - u_0)(x,y) = \begin{cases} \dfrac{x\,r^2}{R^2+r^2}\left[\dfrac{R^2}{x^2+y^2}-1\right], & (x,y) \in B_R(0)\backslash\overline{B_{R/2+r}(0)} \\ 0, & (x,y) \in B_{R/2-r}(0). \end{cases}$$

Example

So the L^2- integral over Ω_r is equal to the L^2 integral over $B_R(0)\backslash\overline{B_{R/2+r}(0)}$. Here, $t = \alpha_1 r = 2r$. Since $r + R/2 < \rho \le R$

$$\frac{1}{2r}\int_{\Omega_r} |w_r|^2\, dx = \frac{r^4}{2r}\int_{\Omega_r^m}\left|\frac{x}{R^2+r^2}\left[\frac{R^2}{x^2+y^2}-1\right]\right|^2 dx \le \frac{r^4}{2r}\left|\frac{5}{R}\right|^2 \pi R^2 \to 0.$$

For the gradient

$$\nabla w_r = \begin{cases} \dfrac{r^2 R^2}{R^2+r^2}\left\{\left[\dfrac{1}{\rho^2}-\dfrac{1}{R^2}\right](1,0) - \dfrac{2x}{\rho^4}(x,y)\right\}, & \text{in } B_R(0)\backslash\overline{B_{R/2+r}(0)} \\ 0, & \text{in } B_{R/2-r}(0). \end{cases}$$

Therefore, for $\rho > R/2 + r$, $(x,y) = \rho(\cos\theta, \sin\theta)$, and $t = \alpha_1 r = 2r$, as $r \to 0$

$$\nabla\left(\frac{w_r}{t^{1/2}}\right) = \frac{r^{3/2}}{\sqrt{2\pi}}\frac{R^2}{R^2+r^2}\left\{\left[\frac{1}{\rho^2}-\frac{1}{R^2}\right](1,0) - \frac{2\cos\theta}{\rho^2}(\cos\theta,\sin\theta)\right\}$$

$$\boxed{\lim_{r\searrow 0}\int_{\Omega_r}\left|\frac{u_r-u_0}{t^{1/2}}\right|^2 + \left\|\nabla\left(\frac{u_r-u_0}{t^{1/2}}\right)\right\|^2 dx = 0 \quad \Rightarrow R(u_0) = 0}$$

$$d_t G(0,u_0) = -\int_E \left(\frac{1}{2}\|\nabla u_0\|^2 - a\,u_0\right) dH^0 = -\frac{1}{2}\|(1,0)\|^2 = -\frac{1}{2}, \quad \boxed{dF(\chi_\Omega;\delta_E) = -\frac{1}{2}}.$$

We give examples in dimension $n = 1$ for $\Omega = (-1, 1)$, $E = \{0\}$, and $t = \alpha_1 r = 2r$, where the polarization term $R(x^0)$ is 0, finite non-zero, and infinite.

$$\left\{\begin{array}{ll} -u_0'' = f, & (-1,1) \\ u_0(\pm 1) = b_\pm & \end{array}\right\} \quad \left\{\begin{array}{ll} -u_r'' = f, & (-1,-r)\cup(r,1) \\ u_r(\pm 1) = b_\pm, & u_r'(\pm r) = 0 \end{array}\right\}. \tag{6.18}$$

The polarization term $R(u_0)$ is the following limit as r goes to zero

$$R(u_0) = -\frac{1}{2}\lim_{r\searrow 0}\frac{u_0'(-r)^2 + u_0'(r)^2}{2r}(1-r) = -\frac{1}{2}\lim_{r\searrow 0}\frac{u_0'(-r)^2 + u_0'(r)^2}{2r} \le 0. \tag{6.19}$$

If $(f\,u_0)(x) = a(x)\,u_0(x)$ is continuous in $(-r, r)$, the t-derivative is also a limit as r goes to zero:

$$d_t G(0, u_0) = -\lim_{r\searrow 0}\frac{1}{2r}\int_{-r}^{r}\frac{1}{2}|u_0'|^2 - f\,u_0\,dx = -\left[\frac{1}{2}|u_0'|^2(0) - (f\,u_0)(0)\right]. \tag{6.20}$$

In view of this simple expression, the polarization term can be controlled by choosing u_0 and computing f and b_-, and b_+.

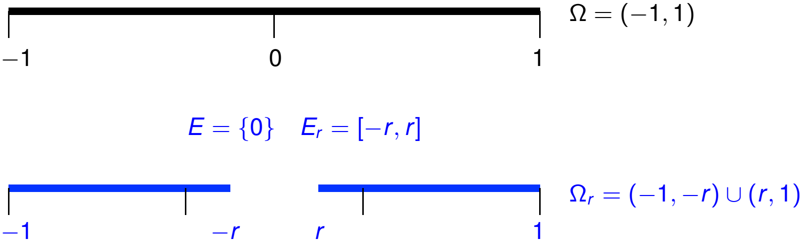


FIGURE 11: Domains Ω and Ω_r

Example

Let $\Omega = (-1,1)$ and $E = \{0\}$.

$$u_0(x) = \frac{1}{2}x^2,\ u_0'(x) = x,\ \left\{\begin{array}{l} f(x) = -1 \\ u_0(\pm 1) = 1/2 \end{array}\right\} \quad \Rightarrow R(u_0) = -\frac{1}{2}\lim_{r\searrow 0}\frac{r^2}{r}(1-r) = 0$$

$$d_t G(0,u_0) = \frac{1}{2}|u_0'|^2(0) - f(0)\,u_0(0) = 0 \quad \Rightarrow \boxed{dF(\chi_\Omega;\delta_E) = 0}.$$

The next example is a variation of the previous one.

Example

Let $\Omega = (-1,1)$ and $E = \{0\}$.

$$u_0(x) = \frac{1}{2}[1-x^2],\ u_0'(x) = -x,\ \left\{\begin{array}{l} f(x) = 1 \\ u_0(\pm 1) = 0 \end{array}\right\} \quad \Rightarrow R(u_0) = -\frac{1}{2}\lim_{r\searrow 0}\frac{r^2}{r}(1-r) = 0 \tag{6.21}$$

$$d_t G(0,u_0) = -\left[\frac{1}{2}|u_0'|^2(0) - f(0)\,u_0(0)\right] = \frac{1}{2} \quad \Rightarrow \boxed{dF(\chi_\Omega;\delta_E) = \frac{1}{2}}. \tag{6.22}$$

Example

Let $\Omega = (-1,1)$ and $E = \{0\}$.

$$u_0(x) = x,\; u_0'(x) = 1,\; \left\{\begin{array}{l} f(x) = 0 \\ u_0(\pm 1) = \pm 1 \end{array}\right\} \quad \Rightarrow R(u_0) = -\frac{1}{2}\lim_{r\searrow 0}\frac{1}{r}(1-r) = -\infty \tag{6.23}$$

$$d_tG(0,u_0) = -\left[\frac{1}{2}|u_0'|^2(0) - f(0)\,u_0(0)\right] = -\frac{1}{2} \quad \Rightarrow \boxed{dF\,(\chi_\Omega;\delta_E) = -\infty} \tag{6.24}$$

In the next example we choose u_0 in such a way that $R(u_0)$ be finite and non-zero:

Example

Let $\Omega = (-1,1)$ and $E = \{0\}$.

$$u_0(x) = \frac{2}{3}|x|^{3/2},\ u_0'(x) = \frac{x}{|x|}|x|^{1/2},\ \left\{ \begin{array}{l} f(x) = -\frac{1}{2}\frac{1}{|x|^{1/2}} \\ u_0(\pm 1) = 2/3 \end{array} \right\} \Rightarrow R(u_0) = -\frac{1}{2},$$

where a is continuous except at $x = 0$ but $a \in L^{2-\varepsilon}(\Omega)$ for all ε, $0 < \varepsilon < 1$. Yet,

$$f(x)\, u_0(x) = -\frac{1}{2}\frac{1}{|x|^{1/2}}\frac{2}{3}|x|^{3/2} = -\frac{1}{3}\,|x|$$

is continuous, and, since u_0' is continuous and $u_0'(0) = 0$, $d_tG(0, u_0) = 0$, and

$$\boxed{dF\,(\chi_\Omega; \delta_E) = -\frac{1}{2}.}$$

So, the continuity of a is not a necessary condition. Moreover,

$$\int_{\Omega_r)} \left| \frac{u_r - u_0}{\sqrt{2r}} \right|^2 dx = \frac{2r}{2r}\,\frac{(1-r)^3}{3} \to \frac{1}{3}, \quad \int_{\Omega_r} \left| \frac{u_r' - u_0'}{\sqrt{2r}} \right|^2 dx = \frac{2r}{2r}\,(1-r) \to 1$$

and $(u_r - u_0)/\sqrt{2r}$ is the restriction of the $H^1(\Omega)$-function $w_0(x) = (1-|x|)/\sqrt{2}$ to Ω_r.

- Thank you for your attention -

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