Model Reduction by Moment Matching for Linear and Nonlinear Time-Delay Systems

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This is a joint work with Alessandro Astolfi

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Contents

▶ Introduction to moment matching
▶ The time domain approach to moment matching
▶ Model reduction for linear time-delay systems
▶ Model reduction for nonlinear time-delay systems
▶ Interpolation at infinitely many points
▶ Model reduction from input/output data
▶ A toolbox for the model reduction by moment matching
▶ Remarks and further research
Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research
Moments - Interpolation approach

\[ \|e\| \leq \beta(\nu) \|u\| \quad \text{with} \quad \lim_{\nu \to n} \beta(\nu) = 0 \]
\[ W(s^*) = W_r(s^*) \quad \ldots \quad \left. \frac{d^k W(s)}{ds^k} \right|_{s=s^*} = \left. \frac{d^k W_r(s)}{ds^k} \right|_{s=s^*} \]
Moments - Interpolation approach

Let

$$V = [(s^* I - A)^{-1} B, (s^* I - A)^{-2} B, \ldots, (s^* I - A)^{-k} B],$$

be the generalized reachability matrix and $W$ any matrix such that

$$W^* V = I$$

Then a reduced order model which matched the moments of the system at $s^*$ is described by the equations

$$\dot{\xi} = W^* A V \xi + W^* B u$$

$$y = C V \xi + D u$$
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- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research
Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
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- Remarks and further research
Consider a linear, single-input, single-output, continuous-time, system described by the equations

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \]  

(1)

and let

\[ W(s) = C(sI - A)^{-1}B \]

be the associated transfer function.

**Definition**

The 0-moment of system (1) at \( s_i \in \mathbb{C} \) is the complex number \( \eta_0(s_i) = C(s_iI - A)^{-1}B \). The \( k \)-moment of system (1) at \( s_i \in \mathbb{C} \) is the complex number

\[ \eta_k(s_i) = \frac{(-1)^k}{k!} \left[ \frac{d^k}{ds^k} (C(sI - A)^{-1}B) \right]_{s=s_i} \]

with \( k \geq 1 \) and integer.
Moments - Time domain approach

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with \( k \geq 1 \) and integer.
Lemma

Suppose \( s_i \notin \sigma(A) \). Then there exists a one-to-one relation between the moments \( \eta_0(s_1), \ldots, \eta_{k_1}(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta}(s_\eta) \) and the matrix \( C\Pi \), where \( \Pi \) is the unique solution of the Sylvester equation

\[
A\Pi + BL = \Pi S,
\]

with \( S \in \mathbb{R}^{\nu \times \nu} \) any non-derogatory matrix with characteristic polynomial

\[
p(s) = \prod_{i=1}^{\eta} (s - s_i)^{k_i}, \text{ where } \nu = \sum_{i=1}^{\eta} k_i,
\]

and \( L \) such that the pair \((L, S)\) is observable.

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Suppose $s_i \notin \sigma(A)$. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \ldots, \eta_{k_1}(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta}(s_\eta)$ and the matrix $C\Pi$, where $\Pi$ is the unique solution of the Sylvester equation

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The interconnected system has a globally invariant manifold given by

\[ \mathcal{M} = \{ (x, \omega) \in \mathbb{R}^{n+\nu} : x = \Pi \omega \} \]

with \( \Pi \) the unique solution of the Sylvester \( A\Pi + BL = \Pi S \).
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with \( \Pi \) the unique solution of the Sylvester \( A\Pi + BL = \Pi S \). As a result

\[ y(t) = C\Pi\omega(t) + Ce^{At}(x(0) - \Pi\omega(0)) \]

where the first term on the right-hand side describes the steady-state response of the system, and the second term on the right-hand side the transient response.
Moments - Time domain approach

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The interconnected system has a local invariant manifold

\[ \mathcal{M} = \{(x, \omega) \in \mathbb{R}^{n+\nu} : x = \pi(\omega)\} \]

if \( \pi(\omega) \) solves \( f(\pi(\omega), l(\omega)) = \frac{\partial \pi}{\partial \omega} s(\omega). \)
The interconnected system has a local *invariant manifold*

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if \( \pi(\omega) \) solves \( f(\pi(\omega), l(\omega)) = \frac{\partial \pi}{\partial \omega} s(\omega) \). Then

\[ y(t) = h(\pi(\omega)) + \varepsilon(t, x(0) - \pi(\omega(0))) \]

and the steady-state response is by definition the moment of the nonlinear system at \( s(\omega) \).
The interconnected system has a local invariant manifold

$$\mathcal{M} = \{(x, \omega) \in \mathbb{R}^{n+\nu} : x = \pi(\omega)\}$$

if $\pi(\omega)$ solves $f(\pi(\omega), l(\omega)) = \frac{\partial \pi}{\partial \omega} s(\omega)$. Then

$$y(t) = h(\pi(\omega)) + \epsilon(t, x(0) - \pi(\omega(0)))$$

and the steady-state response is by definition the moment of the nonlinear system at $s(\omega)$. 
In this talk I will try to convey the message that the one-to-one relation between moments and steady-state response is a flexible and powerful tool to extend the moment matching approach to general class of systems.

Our toolbox is constituted by the steady-state equations

\[ x(t) = \Pi \omega(t) \]

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\[ x(t) = \Pi(t)\omega(t) \]
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- Model reduction for nonlinear time-delay systems
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- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
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Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
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- Time-delay systems are ubiquitous
Motivations

- Time-delay systems are ubiquitous
- Delays generate unexpected behavior
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- Time-delay systems are ubiquitous
- Delays generate unexpected behavior
- Model reduction for linear and nonlinear systems
Motivations

- Time-delay systems are ubiquitous
- Delays generate unexpected behavior
- Model reduction for linear and nonlinear systems
- What is the role of the delay in the reduced order model?
Definition of moment for LTD systems

Consider a linear, single-input, single-output, continuous-time, time-delay system described by the equations

\[ \dot{x} = \sum_{j=0}^{\varsigma} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j}, \quad y = \sum_{j=0}^{\varsigma} C_j x_{\tau_j}, \]  

(2)

and let \( W(s) = \sum_{j=0}^{\varsigma} C_j e^{-s \tau_j} \left( s l - \sum_{j=0}^{\varsigma} A_j e^{-s \tau_j} \right)^{-1} \sum_{j=\varsigma+1}^{\mu} B_j e^{-s \tau_j} \).

**Definition**

The \( k \)-moment of system (2) at \( s_i \in \mathbb{C} \) is the complex number

\[ \eta_k(s_i) = \frac{(-1)^k}{k!} \left[ \frac{d^k}{ds^k} \left( \sum_{j=0}^{\varsigma} C_j e^{-s \tau_j} \left( s l - \sum_{j=0}^{\varsigma} A_j e^{-s \tau_j} \right)^{-1} \sum_{j=\varsigma+1}^{\mu} B_j e^{-s \tau_j} \right) \right]_{s=s_i} \]

with \( k \geq 0 \) integer.
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with \( k \geq 0 \) integer.
 Lemma

Let \( \tilde{A}(s) = \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j} \) and suppose \( s_i \notin \sigma(\tilde{A}(s_i)) \) for all \( i = 1, \ldots, \eta \). Then there exists a one-to-one relation between the moments \( \eta_0(s_1), \ldots, \eta_{k_1}(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta}(s_\eta) \) and the matrix \( \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \), where \( \Pi \) is the unique solution of the Sylvester-like equation

\[
\sum_{j=0}^{\varsigma} A_j \Pi e^{-S\tau_j} - \Pi S = - \sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j}
\]

with \( S \in \mathbb{R}^{\nu \times \nu} \) any non-derogatory matrix with characteristic polynomial \( p(s) = \prod_{i=1}^{\eta} (s - s_i)^{k_i} \), where \( \nu = \sum_{i=1}^{\eta} k_i \) and \( L \) such that the pair \((L, S)\) is observable.
Definition of moment for LTD systems

**Lemma**

Let \( \bar{A}(s) = \sum_{j=0}^{s} A_j e^{-s \tau_j} \) and suppose \( s_i \notin \sigma(\bar{A}(s_i)) \) for all \( i = 1, \ldots, \eta \). Then there exists a one-to-one relation between the moments \( \eta_0(s_1), \ldots, \eta_k(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta}(s_\eta) \) and the matrix \( \sum_{j=0}^{s} C_j \Pi e^{-S \tau_j} \), where \( \Pi \) is the unique solution of the Sylvester-like equation

\[
\sum_{j=0}^{s} A_j \Pi e^{-S \tau_j} - \Pi S = - \sum_{j=s+1}^{\mu} B_j L e^{-S \tau_j}
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with \( S \in \mathbb{R}^{\nu \times \nu} \) any non-derogatory matrix with characteristic polynomial \( p(s) = \prod_{i=1}^{\eta} (s - s_i)^{k_i} \), where \( \nu = \sum_{i=1}^{\eta} k_i \) and \( L \) such that the pair \((L, S)\) is observable.
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with \( S \in \mathbb{R}^{\nu \times \nu} \) any non-derogatory matrix with characteristic polynomial

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\]
The interconnected system has a globally invariant manifold given by

$$\mathcal{M} = \{(x, \omega) \in \mathbb{R}^{n+\nu} : x = \Pi \omega\}$$

with $\Pi$ the unique solution of the Sylvester-like equation.
The interconnected system has a globally **invariant manifold** given by

$$
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\[ \mathcal{M} = \{ (x, \omega) \in \mathbb{R}^{n+\nu} : x = \Pi \omega \} \]

with \( \Pi \) the unique solution of the Sylvester-like equation. As a result

\[
y(t) = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S_j \omega} + \sum_{j=0}^{\varsigma} C_j \mathcal{L}^{-1} \{(sI - \tilde{A}(s))^{-1}(x(0) - \Pi \omega(0))\}
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where the first term on the right-hand side describes the steady-state response of the system, and the second term the transient response.
The interconnected system has a globally invariant manifold given by

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y(t) = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \omega + \sum_{j=0}^{\varsigma} C_j \mathcal{L}^{-1}\left\{ (sl - \tilde{A}(s))^{-1}(x(0) - \Pi \omega(0)) \right\}
\]

where the first term on the right-hand side describes the steady-state response of the system, and the second term the transient response.
\[
\dot{x} = \sum_{j=1}^{q} D_j \dot{x}_c_j + \sum_{j=0}^{s} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j} + \sum_{j=1}^{r} \int_{t-h_j}^{t} (G_j x(\theta) + H_j u(\theta)) d\theta
\]
\[ \dot{x} = \sum_{j=1}^{q} D_j \dot{x}_c_j + \sum_{j=0}^{s} A_j x_{\tau_j} + \sum_{j=s+1}^{\mu} B_j u_{\tau_j} + \sum_{j=1}^{r} \int_{t-h_j}^{t} (G_j x(\theta) + H_j u(\theta)) d\theta \]
Neutral type - Distributed delays

\[ \dot{x} = \sum_{j=1}^{q} D_j \dot{x}_{c_j} + \sum_{j=0}^{s} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j} + \sum_{j=1}^{r} \int_{t-h_j}^{t} (G_j x(\theta) + H_j u(\theta)) d\theta \]

The relation between moments and steady-state response is a powerful tool!

\[ x(t) = \Pi \omega(t) \quad \omega_\tau = e^{S\tau} \omega(t) \quad \int_{t-h}^{t} \omega(\theta) d\theta = S^{-1}(I - e^{-Sh})\omega(t) \]
\[
\dot{x} = \sum_{j=1}^{q} D_j \dot{x}_c_j + \sum_{j=0}^{s} A_j x_{\tau_j} + \sum_{j=s+1}^{\mu} B_j u_{\tau_j} + \sum_{j=1}^{r} \int_{t-h_j}^{t} (G_j x(\theta) + H_j u(\theta)) d\theta
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\]

Hence, the associated Sylvester-like equation is

\[
\sum_{j=0}^{s} A_j \Pi e^{-S\tau_j} + \sum_{j=1}^{r} G_j \Pi S^{-1}(I - e^{-Sh_j}) + \sum_{j=1}^{q} D_j \Pi Se^{-Sc_j} - \Pi S =
\]

\[
= - \sum_{j=s+1}^{\mu} B_j L e^{-S\tau_j} - \sum_{j=1}^{r} H_j LS^{-1}(I - e^{-Sh_j}).
\]

\[\Pi \text{ unique if } s_i \notin \sigma \left( \sum_{j=1}^{q} D_j s e^{-sc_j} + \sum_{j=0}^{s} A_j e^{-s\tau_j} + \sum_{j=1}^{r} G_j \frac{1 - e^{-sh_j}}{s} \right) \text{ and } s_i \neq 0.\]
Neutral type - Distributed delays

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\dot{x} = \sum_{j=1}^{q} D_j \dot{x}_{c_j} + \sum_{j=0}^{s} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j} + \sum_{j=1}^{r} \int_{t-h_j}^{t} (G_j x(\theta) + H_j u(\theta)) d\theta
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The relation between moments and steady-state response is a powerful tool!

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\]

\[
= - \sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j} - \sum_{j=1}^{r} H_j L S^{-1}(I - e^{-Sh_j}).
\]

\(\Pi\) unique if \(s_i \notin \sigma\left(\sum_{j=1}^{q} D_j se^{-sc_j} + \sum_{j=0}^{s} A_j e^{-s\tau_j} + \sum_{j=1}^{r} G_j \frac{1 - e^{-sh_j}}{s}\right)\) and \(s_i \neq 0\).
Reduced model with free parameters

The system

\[ \dot{\xi} = \sum_{j=0}^{\rho} F_j \xi_{\chi_j} + \sum_{j=\rho+1}^{\rho} G_j u_{\chi_j}, \quad \psi = \sum_{j=0}^{d} H_j \xi_{\chi_j}, \]

is a model of the original system at \( S \), if \( s_l \notin \sigma \left( \sum_{j=0}^{\rho} F_j e^{-s_l \chi_j} \right) \) for all \( l = 1, \ldots, \eta \), and there exists a unique solution \( P \) of the equation

\[ \sum_{j=0}^{\rho} F_j Pe^{-S\chi_j} - PS = - \sum_{j=\rho+1}^{\rho} G_j Le^{-S\chi_j}, \]

such that

\[ \sum_{j=0}^{\delta} C_j \Pi e^{-S\tau_j} = \sum_{j=0}^{d} H_j Pe^{-S\chi_j} \]
The system

\[
\dot{\xi} = \sum_{j=0}^{\varrho} F_j \xi_j + \sum_{j=\varrho+1}^{\rho} G_j u \chi_j, \quad \psi = \sum_{j=0}^{d} H_j \xi_j,
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\[
\sum_{j=0}^{\varrho} F_j P e^{-S \chi_j} - PS = -\sum_{j=\varrho+1}^{\rho} G_j L e^{-S \chi_j},
\]

such that

\[
\sum_{j=0}^{\varsigma} C_j \Pi e^{-S \tau_j} = \sum_{j=0}^{d} H_j P e^{-S \chi_j}.
\]
The system

\[ \dot{\xi} = \sum_{j=0}^{\varrho} F_j \xi_{\chi_j} + \sum_{j=\varrho+1}^{\rho} G_j u_{\chi_j}, \quad \psi = \sum_{j=0}^{d} H_j \xi_{\chi_j}, \]

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such that

\[ \sum_{j=0}^{\varsigma} C_j \Pi e^{-S \tau_j} = \sum_{j=0}^{d} H_j Pe^{-S \chi_j}. \]
To construct a family of models that achieves moment matching at $\nu$ points select $P = I$. This yields the family of reduced order models

\[
\dot{\xi} = \left( S - \sum_{j=\rho+1}^{\rho} G_j L e^{-S\chi_j} - \sum_{j=1}^{\rho} F_j e^{-S\chi_j} \right) \xi + \sum_{j=1}^{\rho} F_j \xi \chi_j + \sum_{j=\rho+1}^{\rho} G_j u \chi_j,
\]

\[
\psi = \left( \sum_{j=0}^{\gamma} C_j \Pi e^{-S\tau_j} - \sum_{j=1}^{d} H_j e^{-S\chi_j} \right) \xi + \sum_{j=1}^{d} H_j \xi \chi_j,
\]

with $G_j$, $F_j$ and $H_j$ any matrices.
To construct a family of models that achieves moment matching at \( \nu \) points select \( P = I \). This yields the family of reduced order models

\[
\begin{align*}
\dot{\xi} &= \left( S - \sum_{j=\rho+1}^{\rho} G_j L e^{-S \chi_j} - \sum_{j=1}^{\rho} F_j e^{-S \chi_j} \right) \xi + \sum_{j=1}^{\rho} F_j \xi_{\chi_j} + \sum_{j=\rho+1}^{\rho} G_j u_{\chi_j}, \\
\psi &= \left( \sum_{j=0}^{S} C_j \Pi e^{-S \tau_j} - \sum_{j=1}^{d} H_j e^{-S \chi_j} \right) \xi + \sum_{j=1}^{d} H_j \xi_{\chi_j},
\end{align*}
\]

with \( G_j, F_j \) and \( H_j \) any matrices.
To construct a family of models that achieves moment matching at $\nu$ points select $P = I$. This yields the family of reduced order models

$$
\dot{\xi} = \left( \sum_{j=\rho+1}^{\rho} G_j L e^{-S \chi_j} - \sum_{j=1}^{\rho} F_j e^{-S \chi_j} \right) \xi + \sum_{j=1}^{\rho} F_j \dot{\xi} \chi_j + \sum_{j=\rho+1}^{\rho} G_j u \chi_j,
$$

$$
\psi = \left( \sum_{j=0}^{\varsigma} C_j \Pi e^{-S \tau_j} - \sum_{j=1}^{d} H_j e^{-S \chi_j} \right) \xi + \sum_{j=1}^{d} H_j \xi \chi_j,
$$

with $G_j$, $F_j$ and $H_j$ any matrices.

The delay-free model is in this family

$$
\dot{\xi} = (S - G_1 L) \xi + Gu,
$$

$$
\psi = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S \tau_j} \xi
$$
Consider the model of a LC transmission line described by the equations

\[ \dot{x}_1 = -\frac{1}{C_1} \left( \frac{1}{R_1} + \sqrt{\frac{C_0}{L}} \right) x_1 - \frac{2}{C_1} \sqrt{\frac{C_0}{L}} \frac{1 - R_0 \sqrt{\frac{C_0}{L}}}{1 + R_0 \sqrt{\frac{C_0}{L}}} x_{2\tau} + b_1 u, \]

\[ \dot{x}_2 = x_1 + \frac{1 - R_0 \sqrt{\frac{C_0}{L}}}{1 + R_0 \sqrt{\frac{C_0}{L}}} x_{2\tau} + b_1 u, \]

\[ y = c_1 x_1 + c_2 x_2, \]
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A family of reduced order models at \((S = 1, L = 1)\), parameterized in \(G\), is described by the equations

\[
\dot{\xi} = \left( 1 - e^{-\tau} \frac{1 - R_0 \sqrt{\frac{C_0}{L}}}{1 + R_0 \sqrt{\frac{C_0}{L}}} - G \right) \xi + \frac{1 - R_0 \sqrt{\frac{C_0}{L}}}{1 + R_0 \sqrt{\frac{C_0}{L}}} \xi_\tau + Gu,
\]

\[
\psi = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \Pi \xi.
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Interpolating $(\rho + 1)\nu$ points

Can we exploit the additional free parameters to interpolate more points?
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Let \( S_a \in \mathbb{R}^{\nu \times \nu} \) and \( S_b \in \mathbb{R}^{\nu \times \nu} \) be two matrices such that \( \sigma(S_a) \cap \sigma(S_b) = \emptyset \).

Consider \( F_0 \) and \( H_0 \) given before with \( \chi_2 = 0, S = S_a, d = 1 \) and \( L = L_a = L_b \). Then the selection

\[
F_1 = (S_b - S_a - G_3(e^{-S_b \chi_3} - e^{-S_a \chi_3}))(e^{-S_b \chi_1} - e^{-S_a \chi_1})^{-1},
\]

\[
F_0 = S_a - G_2 L - G_3 L e^{-S_a \chi_3} - F_1 e^{-S_a \chi_1},
\]

\[
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\[
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belongs to the family of reduced order models achieving moment matching at \( S_a \) and \( S_b \), for any \( G_2 \) and \( G_3 \), with \( P_a = P_b = I \).
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Interpolating \((\rho + 1)\nu\) points

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Bode plot of a $n = 1006$ delay-free system (blue/solid line), of a $\nu = 8$ delay-free reduced order model (black/dash-dotted line) and a $\nu = 8$ time-delay reduced order model (red/dotted line). The squares indicate the first set of interpolation points, whereas the circles indicate the second set.
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Contents

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Moment for NLTD systems

Consider a nonlinear, single-input, single-output, continuous-time, time-delay system described by the equations

\[ \dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_\zeta}, u_{\tau_\mu}), \quad y = h(x) \]

Consider a signal generator described by the equations

\[ \dot{\omega} = s(\omega), \quad \theta = l(\omega), \]

and the interconnected system

\[ \dot{\omega} = s(\omega), \quad \dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_\zeta}, l(\omega_{\tau_\mu})), \quad y = h(x). \]

Assumption

There exists a unique mapping \( \pi(\omega) \), locally defined in a neighborhood of \( \omega = 0 \), which solves the partial differential equation

\[ \frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\bar{\omega}_{\tau_0}), \ldots, \pi(\bar{\omega}_{\tau_\zeta}), l(\bar{\omega}_{\tau_\mu})) \] (3)

where \( \bar{\omega}_{\tau_i} = \Phi_{\tau_i}^s(\omega) \) is the flow of the vector field \( s \) at \( \tau_i \).
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### Assumption

*The signal generator is observable.*

### Definition

The function $h(\pi(\omega))$, with $\pi$ solution of equation (3), is the *moment of the system at* $(s(\omega), l(\omega))$.

### Theorem

*Assume the zero equilibrium of the system $\dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_\varsigma}, 0)$ is locally exponentially stable and $s(\omega)$ is Poisson stable. Then there exists a unique $\pi(\omega)$ and the moment of the system at $(s(\omega), l(\omega))$ coincides with the steady-state response of the output of the interconnected system.*
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A family of models that achieves moment matching at \((s(\omega), l(\omega))\) is described by the equations

\[
\dot{\xi} = s(\xi) - \delta(\xi)l(\bar{\xi}_\chi u) - \gamma(\bar{\xi}_\chi_1, \ldots, \bar{\xi}_\chi e) + \gamma(\xi_\chi_1, \ldots, \xi_\chi e) + \delta(\xi)u_\chi u
\]

\[
\psi = h(\pi(\xi))
\]

where \(\bar{\omega}_\chi_i = \Phi^s_{\chi i}(\omega)\) and \(\delta\) and \(\gamma\) are arbitrary mappings such that

\[
\frac{\partial p}{\partial \omega} s(\omega) = s(p(\omega)) - \delta(p(\omega))l(p(\bar{\omega}_\chi u)) + \delta(p(\omega))l(\omega_\chi u) - \\
- \gamma(p(\bar{\omega}_\chi_1), \ldots, p(\bar{\omega}_\chi e)) + \gamma(p(\omega_\chi_1), \ldots, p(\omega_\chi e))
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has the unique solution \(p(\omega) = \omega\).
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\]

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\[
\frac{\partial^2 \theta}{\partial z^2}(z, t) = \frac{l}{GJ \partial t^2}(z, t), \quad z \in (0, L), \; t > 0
\]

coupled to the mixed boundary conditions

\[
GJ \frac{\partial \theta}{\partial z}(0, t) = c_a \left( \frac{\partial \theta}{\partial z}(0, t) - \Omega(t) \right), \quad GJ \frac{\partial \theta}{\partial z}(L, t) + l_B \frac{\partial^2 \theta}{\partial t^2}(L, t) = -T \left( \frac{\partial \theta}{\partial t}(L, t) \right)
\]
Reduced order model

\[ \dot{\xi} = -\delta(\xi) [\xi - r\tau_2] \]

\[ \psi = \pi(\xi) \]

Open-loop reduced order model

\[ \dot{\xi} = -\delta(\xi) [\xi - \mu]\tau_2] \]

\[ \mu = -k_1\pi(\dot{\xi}\tau_2) - k_2\pi(\xi\tau_2) + r \]

\[ \psi = \pi(\xi) \]
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The results described are based on the availability of a differential representation of the signal generator, namely $\dot{\omega} = S\omega$. However, there are notable applications in which this may not be the case. For instance, the input of a dynamical system describing a power electronic device can often be a PWM wave (e.g. a square or sawtooth wave) which cannot be represented as the output of a system described by smooth differential equations.
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Consider a square wave $\nabla(t)$ defined as

$$\nabla(t) = \text{sign}(\sin(t)) = \begin{cases} 
1, & (k-1)\pi < t < k\pi, \\
0, & t = k\pi \text{ or } t = (k+1)\pi, \\
-1, & k\pi < t < (k+1)\pi,
\end{cases}$$

i.e. with $\text{sign}(0) = 0$, and $k = 1, 3, 5, \ldots, +\infty$. 
Consider a square wave \( \square(t) \) defined as

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\end{cases}
\]

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The Laplace transform of this function is

\[
\mathcal{L}(\square(t)) = \frac{1 - e^{-s\pi}}{s(1 + e^{-s\pi})},
\]

and this has the poles

\[
s_1 = 0, \quad s_i = (2j + 1)i,
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Since the function $\Box(t)$ is periodic, it admits a Fourier series, namely

$$\Box(t) = \frac{4}{\pi} \sum_{i=1,3,5,\ldots,+\infty} \frac{1}{i} \sin(it).$$
Analysis of a square wave

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The Laplace and Fourier transform of the square wave suggest that we could describe $\square(t)$ by means of the infinite dimensional system

$$\dot{\omega} = \begin{bmatrix}
... & ... & ... & ... \\
... & 0 & +i & 0 \\
0 & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -2i & 0 & 0 \\
... & ... & ... & ...
\end{bmatrix} \omega$$

with output $\square = P\omega$ for some “matrix” $P$. 
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To overcome these issues we consider signal generators in explicit form. Thus, consider

\[ \omega(t) = \Lambda(t, t_0)\omega_0, \quad u = L\omega, \]

Note that for linear systems in implicit form

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But it describes a very large class of signals: noncontinuous periodic signals, time-varying systems, a subclass of hybrid systems, a subclass of nonlinear systems,...

We want to characterize the “moments” of the following interconnection
\[ \omega(t) = \Lambda(t, t_0)\omega_0 \]
\[ \dot{x} = Ax + BL\omega \]
\[ y = Cx \]
Characterization of the moments

**Theorem**

Let \( \Pi(t) = \left( e^{A(t-t_0)} \Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B L \Lambda(\tau, t_0) d\tau \right) \Lambda(t, t_0)^{-1} \)

be a family of matrix valued functions parametrized in \( \Pi(t_0) \in \mathbb{R}^{n \times \nu} \). Given "mild" assumptions there exists a unique \( \Pi_\infty(t_0) \) such that, for any \( \Pi(t_0) \),

\[
\lim_{t \to +\infty} \Pi(t) - \Pi_\infty(t) = 0.
\]

Moreover, if \( x(t_0) = \Pi_\infty(t_0) \omega(t_0) \) then \( x(t) - \Pi_\infty(t) \omega(t) = 0 \) for all \( t \geq t_0 \), and the set \( \{(x, \omega) | x(t) = \Pi_\infty(t) \omega(t)\} \) is attractive.
Theorem

Let \( \Pi(t) = \left( e^{A(t-t_0)} \Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B L \Lambda(\tau, t_0) d\tau \right) \Lambda(t, t_0)^{-1} \)

be a family of matrix valued functions parametrized in \( \Pi(t_0) \in \mathbb{R}^{n \times \nu} \). Given “mild” assumptions there exists a unique \( \Pi_\infty(t_0) \) such that, for any \( \Pi(t_0) \),

\[
\lim_{t \to +\infty} \Pi(t) - \Pi_\infty(t) = 0.
\]

Moreover, if \( x(t_0) = \Pi_\infty(t_0) \omega(t_0) \) then

\[
x(t) - \Pi_\infty(t) \omega(t) = 0 \text{ for all } t \geq t_0, \text{ and the set } \{ (x, \omega) \mid x(t) = \Pi_\infty(t) \omega(t) \} \text{ is attractive.}\]
Characterization of the moments

**Theorem**

Let

\[ \Pi(t) = \left( e^{A(t-t_0)} \Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B L \Lambda(\tau, t_0) d\tau \right) \Lambda(t, t_0)^{-1} \]

be a family of matrix valued functions parametrized in \( \Pi(t_0) \in \mathbb{R}^{n \times \nu} \). Given "mild" assumptions there exists a unique \( \Pi_\infty(t_0) \) such that, for any \( \Pi(t_0) \),

\[ \lim_{t \to +\infty} \Pi(t) - \Pi_\infty(t) = 0. \]

Moreover, if \( x(t_0) = \Pi_\infty(t_0) \omega(t_0) \) then

\[ x(t) - \Pi_\infty(t) \omega(t) = 0 \text{ for all } t \geq t_0, \text{ and the set } \{ (x, \omega) | x(t) = \Pi_\infty(t) \omega(t) \} \text{ is attractive.} \]
Theorem

Let \( \Pi(t) = \left( e^{A(t-t_0)} \Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B L \Lambda(\tau, t_0) d\tau \right) \Lambda(t, t_0)^{-1} \) be a family of matrix valued functions parametrized in \( \Pi(t_0) \in \mathbb{R}^{n \times \nu} \). Given “mild” assumptions there exists a unique \( \Pi_\infty(t_0) \) such that, for any \( \Pi(t_0) \), \( \lim_{t \to +\infty} \Pi(t) - \Pi_\infty(t) = 0 \). Moreover, if \( x(t_0) = \Pi_\infty(t_0) \omega(t_0) \) then \( x(t) - \Pi_\infty(t) \omega(t) = 0 \) for all \( t \geq t_0 \), and the set \( \{ (x, \omega) \mid x(t) = \Pi_\infty(t) \omega(t) \} \) is attractive.
Characterization of the moments

**Theorem**

Let $\Pi(t) = \left(e^{A(t-t_0)}\Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}BL\Lambda(\tau, t_0)d\tau\right)\Lambda(t, t_0)^{-1}$

be a family of matrix valued functions parametrized in $\Pi(t_0) \in \mathbb{R}^{n \times \nu}$. Given “mild” assumptions there exists a unique $\Pi_\infty(t_0)$ such that, for any $\Pi(t_0)$, $\lim_{t \to +\infty} \Pi(t) - \Pi_\infty(t) = 0$. Moreover, if $x(t_0) = \Pi_\infty(t_0)\omega(t_0)$ then $x(t) - \Pi_\infty(t)\omega(t) = 0$ for all $t \geq t_0$, and the set $\{(x, \omega) | x(t) = \Pi_\infty(t)\omega(t)\}$ is attractive.

**Remark**

$\Pi_\infty(t)$ is also the unique solution of

$$\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t)\dot{\Lambda}(t, t_0)\Lambda(t, t_0)^{-1}$$

with the initial condition $\Pi(t_0) = \Pi_\infty(t_0)$. From a practical point of view, it is necessary to know the initial condition $\Pi_\infty(t_0)$. However, since the motion $\Pi_\infty(t)$ is attractive, any solution of the two equations converges to $\Pi_\infty(t)$. 

Giordano Scarciotti
Model Reduction by Moment Matching for Linear and Nonlinear Time-Delay Systems
33/56
Theorem

Let \( \Pi(t) = \left( e^{A(t-t_0)} \Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B L \Lambda(\tau, t_0) d\tau \right) \Lambda(t, t_0)^{-1} \)

be a family of matrix valued functions parametrized in \( \Pi(t_0) \in \mathbb{R}^{n \times \nu} \). Given “mild” assumptions there exists a unique \( \Pi_{\infty}(t_0) \) such that, for any \( \Pi(t_0) \),

\[
\lim_{t \to +\infty} \Pi(t) - \Pi_{\infty}(t) = 0.
\]

Moreover, if \( x(t_0) = \Pi_{\infty}(t_0) \omega(t_0) \) then \( x(t) - \Pi_{\infty}(t) \omega(t) = 0 \) for all \( t \geq t_0 \), and the set \( \{ (x, \omega) \mid x(t) = \Pi_{\infty}(t) \omega(t) \} \) is attractive.

Remark

\( \Pi_{\infty}(t) \) is also the unique solution of

\[
\dot{\Pi}(t) = A \Pi(t) + B L - \Pi(t) \dot{\Lambda}(t, t_0) \Lambda(t, t_0)^{-1}
\]

with the initial condition \( \Pi(t_0) = \Pi_{\infty}(t_0) \). From a practical point of view, it is necessary to know the initial condition \( \Pi_{\infty}(t_0) \). However, since the motion \( \Pi_{\infty}(t) \) is attractive, any solution of the two equations converges to \( \Pi_{\infty}(t) \).
Consider the signal generator

$$\omega(t) = \omega(t - T),$$
$$\omega(t) = h(t, t_0)\omega_0, \quad t_0 - T \leq t < t_0,$$
$$u = L\omega,$$

then $\Pi_\infty(t)$ becomes

$$\Pi_\infty(t) = (I - e^{AT})^{-1} \left[ \int_{t-T}^{t} e^{A(t-\tau)} B L \Lambda(\tau, t_0) d\tau \right] \Lambda(t, t_0)^{-1}$$
Consider the signal generator

\[ \omega(t) = \omega(t - T), \]
\[ \omega(t) = h(t, t_0)\omega_0, \quad t_0 - T \leq t < t_0, \]
\[ u = L\omega, \]

then \( \Pi_\infty(t) \) becomes

\[ \Pi_\infty(t) = (I - e^{AT})^{-1} \left[ \int_{t-T}^{t} e^{A(t-\tau)}BL\Lambda(\tau, t_0)d\tau \right] \Lambda(t, t_0)^{-1} \]
A numerical example

Consider the matrix of square waves

\[
\Lambda_\cap(t, 0) = \begin{bmatrix}
\cap \left( \frac{2\pi}{T} t + \frac{\pi}{2} \right) & -\cap \left( \frac{2\pi}{T} t \right) \\
\cap \left( \frac{2\pi}{T} t \right) & \cap \left( \frac{2\pi}{T} t + \frac{\pi}{2} \right)
\end{bmatrix}.
\]

The previous equation computed for \( t = 0 \)

\[
\Pi_\infty(0) = -A^{-1}(I - e^{AT})^{-1} \left[ (e^{\frac{3}{4}AT} - e^{AT} + e^{\frac{1}{2}AT} - e^{\frac{1}{4}AT}) BL + \right.
\]

\[
+ \left( e^{\frac{1}{2}AT} - e^{\frac{3}{4}AT} + e^{\frac{1}{4}AT} - I \right) BL\Lambda_\cap \left( \frac{T}{4}, 0 \right)
\]

\[
+ \left. \left( e^{\frac{1}{2}AT} - e^{\frac{3}{4}AT} + e^{\frac{1}{4}AT} - I \right) BL\Lambda_\cap \left( \frac{T}{4}, 0 \right) \right]
\]
Looking at new $\Pi$'s

\[
\Lambda_{\sim}(t, 0) = \begin{bmatrix}
\cos\left(\frac{2\pi}{T} t\right) & -\sin\left(\frac{2\pi}{T} t\right) \\
\sin\left(\frac{2\pi}{T} t\right) & \cos\left(\frac{2\pi}{T} t\right)
\end{bmatrix}
\]

\[
\Lambda_{\land}(t, 0) = \begin{bmatrix}
\wedge\left(\frac{2\pi}{T} t + \frac{\pi}{2}\right) & -\wedge\left(\frac{2\pi}{T} t\right) \\
\wedge\left(\frac{2\pi}{T} t\right) & \wedge\left(\frac{2\pi}{T} t + \frac{\pi}{2}\right)
\end{bmatrix}
\]

Time history of the entries of the matrices $\Pi_{\sim}$ (top), $\Pi_{\land}$ (middle) and $\Pi_{\nabla}$ (bottom).

Time history of the output (solid lines) $y_{\sim}$ (top), $y_{\land}$ (middle) and $y_{\nabla}$ (bottom). Time histories of the steady-state of the output (dotted lines) computed as $C\Pi_{\sim}\omega$, $C\Pi_{\land}\omega$ and $C\Pi_{\nabla}\omega$. 

Giordano Scarciotti  
Model Reduction by Moment Matching for Linear and Nonlinear Time-Delay Systems  
36/56
A new family of reduced order models

**Definition**

The system described by the equations

\[ \xi(t) = F(t, t_0)\xi_0 + \int_{t_0}^{t} G(t - \tau)u(\tau)d\tau, \]

\[ \psi(t) = H(t)\xi(t), \]

is a *model of the system*, if there exists a unique solution \( P_\infty(t) \) of the equation

\[ P(t) = \left( F(t, t_0)P(t_0) + \int_{t_0}^{t} G(t - \tau)L\Lambda(\tau, t_0)d\tau \right) \Lambda^{-1}(t, t_0) \]

with \( P(t_0) = P_\infty(t_0) \) such that for any \( P(t_0) \), \( \lim_{t \to +\infty} P(t) - P_\infty(t) = 0 \) and

\[ C\Pi_\infty(t) = H(t)P_\infty(t) \]
A new family of reduced order models

Definition

The system described by the equations

\[ \xi(t) = F(t, t_0)\xi_0 + \int_{t_0}^{t} G(t - \tau)u(\tau)\,d\tau, \]
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is a model of the system, if there exists a unique solution \( P(\infty)(t) \) of the equation

\[ P(t) = \left( F(t, t_0)P(t_0) + \int_{t_0}^{t} G(t - \tau)L\Lambda(\tau, t_0)\,d\tau \right)\Lambda^{-1}(t, t_0) \]

with \( P(t_0) = P(\infty)(t_0) \) such that for any \( P(t_0) \), \( \lim_{t \rightarrow +\infty} P(t) - P(\infty)(t) = 0 \) and

\[ C\Pi(\infty)(t) = H(t)P(\infty)(t) \]
A new family of reduced order models

**Definition**

The system described by the equations

\[
\begin{align*}
\xi(t) &= F(t, t_0)\xi_0 + \int_{t_0}^{t} G(t - \tau)u(\tau)d\tau, \\
\psi(t) &= H(t)\xi(t),
\end{align*}
\]

is a *model of the system*, if there exists a unique solution \( P_\infty(t) \) of the equation

\[
P(t) = \left( F(t, t_0)P(t_0) + \int_{t_0}^{t} G(t - \tau)L\Lambda(\tau, t_0)d\tau \right)\Lambda^{-1}(t, t_0)
\]

with \( P(t_0) = P_\infty(t_0) \) such that for any \( P(t_0) \), \( \lim_{t \to +\infty} P(t) - P_\infty(t) = 0 \) and

\[
C\Pi_\infty(t) = H(t)P_\infty(t)
\]
The periodic family

Definition

The system

\[ \dot{\xi} = \tilde{F}\xi + \tilde{G}u, \]

\[ \psi(t) = C\Pi_\infty(t)P_\infty(t)^{-1}\xi(t), \]

is a model of the system, if \( \sigma(\tilde{F}) \in \mathbb{C}_{<0} \) and

\[ P_\infty(t) = (I - e^{\tilde{F}T})^{-1} \left[ \int_{t-T}^{t} e^{\tilde{F}(t-\tau)}\tilde{G}L\Lambda(\tau, t_0) d\tau \right] \Lambda(t, t_0)^{-1}, \]

is non-singular for all \( t \in \mathbb{R}_{\geq 0} \).
Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research
Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research
If we have the steady-state response $C\Pi\omega(t)$, how do we recover the moments $\Pi$?
If we have the steady-state response $C\Pi\omega(t)$, how do we recover the moments $\Pi$?

How do we obtain a reduced order model if we do not have the matrices $A$, $B$, $C$, but we have measurements of the input and output of the system?
Let’s manipulate the response

Recall that the output of a linear system can be written as

\[ y(t) = C\Pi \omega(t) + C e^{At}(x(0) - \Pi \omega(0)) \]

This can be rewritten as

\[ \text{vec}(C\Pi \omega(t)) - \text{vec}(Ce^{At}\Pi \omega(0)) = \text{vec}(y(t) - Ce^{At}x(0)), \]

and

\[ (\omega(t)^T \otimes C - \omega(0)^T \otimes Ce^{At}) \text{vec}(\Pi) = \text{vec}(y(t) - Ce^{At}x(0)). \]

Finally

\[ (\omega(0)^T \otimes C)(e^{S^T}t \otimes I - I \otimes e^{At}) \text{vec}(\Pi) = \text{vec}(y(t) - Ce^{At}x(0)) \]
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and

$$(\omega(t)^\top \otimes C - \omega(0)^\top \otimes Ce^{At}) \text{vec}(\Pi) = \text{vec}(y(t) - Ce^{At}x(0)).$$

Finally

$$(\omega(0)^\top \otimes C)(e^{S^T}t \otimes I - I \otimes e^{At}) \text{vec}(\Pi) = \text{vec}(y(t) - Ce^{At}x(0))$$
Let’s manipulate the response

Define the time-snapshots $R_k \in \mathbb{R}^{w \times n\nu}$ and $\Upsilon_k \in \mathbb{R}^w$ as

$$\begin{align*}
R_k &= \begin{bmatrix}
(\omega(0)^{\top} \otimes C)(e^{S^{\top}t_{k-w+1}} \otimes I - I \otimes e^{At_{k-w+1}}) \\
\vdots \\
(\omega(0)^{\top} \otimes C)(e^{S^{\top}t_{k-1}} \otimes I - I \otimes e^{At_{k-1}}) \\
(\omega(0)^{\top} \otimes C)(e^{S^{\top}t_k} \otimes I - I \otimes e^{At_k})
\end{bmatrix}, \\
\Upsilon_k &= \begin{bmatrix}
y(t_{k-w+1}) - Ce^{At_{k-w+1}}x(0) \\
\vdots \\
y(t_{k-1}) - Ce^{At_{k-1}}x(0) \\
y(t_k) - Ce^{At_k}x(0)
\end{bmatrix}.
\end{align*}$$

This yields the on-line estimate

$$\text{vec}(\Pi_k) = (R_k^{\top} R_k)^{-1} R_k^{\top} \Upsilon_k$$
Let's manipulate the response

Define the time-snapshots $R_k \in \mathbb{R}^{w \times n\nu}$ and $\Upsilon_k \in \mathbb{R}^w$ as

\[
R_k = \begin{bmatrix}
(\omega(0)^\top \otimes C) \left( e^{S^\top t_{k-w+1}} \otimes I - I \otimes e^{A t_{k-w+1}} \right) \\
\vdots \\
(\omega(0)^\top \otimes C) \left( e^{S^\top t_{k-1}} \otimes I - I \otimes e^{A t_{k-1}} \right) \\
(\omega(0)^\top \otimes C) \left( e^{S^\top t_{k}} \otimes I - I \otimes e^{A t_{k}} \right)
\end{bmatrix},
\]

\[
\Upsilon_k = \begin{bmatrix}
y(t_{k-w+1}) - C e^{A t_{k-w+1}} x(0) \\
\vdots \\
y(t_{k-1}) - C e^{A t_{k-1}} x(0) \\
y(t_{k}) - C e^{A t_{k}} x(0)
\end{bmatrix}.
\]

This yields the on-line estimate

\[
\text{vec}(\Pi_k) = (R_k^\top R_k)^{-1} R_k^\top \Upsilon_k
\]
Exploiting the steady-state

Note that the equation can be written as

\[ y(t) = C \Pi \omega(t) + \varepsilon(t), \]

with \( \varepsilon(t) = Ce^{At}(x(0) - \Pi \omega(0)) \) an exponentially decaying signal.
Note that the equation can be written as

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Note that the equation can be written as

\[ y(t) = C\Pi \omega(t) + \varepsilon(t), \]

with \( \varepsilon(t) = Ce^{At}(x(0) - \Pi \omega(0)) \) an exponentially decaying signal.

Thus, let \( \widetilde{C\Pi} \) be such that

\[ y(t) = \widetilde{C\Pi} \omega(t), \]

and define the time-snapshots \( \tilde{R}_k \in \mathbb{R}^{w \times \nu} \) and \( \tilde{\Upsilon}_k \in \mathbb{R}^w \) as

\[ \tilde{R}_k = \begin{bmatrix} \omega(t_{k-w+1}) & \ldots & \omega(t_{k-1}) & \omega(t_k) \end{bmatrix}^T \]

and

\[ \tilde{\Upsilon}_k = \begin{bmatrix} y(t_{k-w+1}) & \ldots & y(t_{k-1}) & y(t_k) \end{bmatrix}^T. \]

Then

\[ \text{vec}(\widetilde{C\Pi}_k) = (\tilde{R}_k^T \tilde{R}_k)^{-1}\tilde{R}_k^T\tilde{\Upsilon}_k, \]

is an approximation of the on-line estimate \( C\Pi_k \).
A recursive implementation

It is easy to derive a recursive least-squares estimation of \( \tilde{C}\Pi_k \). To this end, let

\[
\Phi_k = \left( \tilde{R}_k^\top \tilde{R}_k \right)^{-1},
\]
\[
\Psi_k = \left( \tilde{R}_{k-1}^\top \tilde{R}_{k-1} + \omega(t_k)\omega(t_k)^\top \right)^{-1}.
\]

Then

\[
\tilde{C}\Pi_k = \tilde{C}\Pi_{k-1} + \Phi_k \omega(t_k)(y(t_k) - \omega(t_k)^\top \tilde{C}\Pi_{k-1})
\]
\[
-\Phi_k \omega(t_{k-w})(y(t_{k-w}) - \omega(t_{k-w})^\top \tilde{C}\Pi_{k-1}),
\]

with

\[
\Phi_k = \Psi_k - \Psi_k \omega(t_{k-w}) \times
\]
\[
(l + \omega(t_{k-w})^\top \Psi_k \omega(t_{k-w}))^{-1} \omega(t_{k-w})^\top \Psi_k
\]

and

\[
\Psi_k = \Phi_{k-1} - \Phi_{k-1} \omega(t_k) \times
\]
\[
(l + \omega(t_k)^\top \Phi_{k-1} \omega(t_k))^{-1} \omega(t_k)^\top \Phi_{k-1}.
\]

For SISO systems the two matrix inversions are two divisions. The computation complexity of updating the estimate is \( \mathcal{O}(1) \).
A family of reduced order models

Definition

The system described by the equations

\[ \dot{\xi} = F_k \xi + G_k u, \quad \phi = H_k \xi, \]

is a model of the system at \((S,L)\) at time \(t_k\), if there exists a unique solution \(P_k\) of the equation

\[ F_k P_k + G_k L = P_k S, \]

such that

\[ \tilde{C} \tilde{\Pi}_k = H_k P_k, \]

Remark

Select \(P_k = I\), for all \(k \geq 0\). If \(\sigma(F_k) \cap \sigma(S) = \emptyset\) for all \(k \geq 0\), then the model

\[ \dot{\xi} = (S - G_k L)\xi + G_k u, \]

\[ \phi = \tilde{C} \tilde{\Pi}_k \xi, \]

is a model of the system at \((S,L)\) at time \(t_k\).
A family of reduced order models

### Definition

The system described by the equations

\[
\dot{\xi} = F_k \xi + G_k u, \quad \phi = H_k \xi,
\]

is a *model of the system at* \((S,L)\) *at time* \(t_k\), if there exists a unique solution \(P_k\) of the equation

\[
F_k P_k + G_k L = P_k S,
\]

such that

\[
\tilde{C} \Pi_k = H_k P_k,
\]

### Remark

*Select* \(P_k = I\), *for all* \(k \geq 0\). *If* \(\sigma(F_k) \cap \sigma(S) = \emptyset\) *for all* \(k \geq 0\), *then the model*

\[
\dot{\xi} = (S - G_k L) \xi + G_k u, \\
\phi = \tilde{C} \Pi_k \xi,
\]

*is a model of the system at* \((S,L)\) *at time* \(t_k\).*
Linear time-delay systems

These results can be easily extended to linear time-delay systems. In fact, we have already seen that for linear time-delay systems the following holds

\[ y(t) = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \omega(t) + \varepsilon(t), \]

Then

\[
\text{vec} \left( \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \right) = (\tilde{R}_k^\top \tilde{R}_k)^{-1} \tilde{R}_k^\top \tilde{\Upsilon}_k,
\]

is an approximation of the on-line estimate \( \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \), and families of reduced order models at time \( t_k \) can be easily defined.
These results can be easily extended to linear time-delay systems. In fact, we have already seen that for linear time-delay systems the following holds:

$$y(t) = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \omega(t) + \varepsilon(t),$$

Then

$$\text{vec} \left( \sum_{j=0}^{\varsigma} C_j \Pi e_k^{-S\tau_j} \right) = (\tilde{R}_k^\top \tilde{R}_k)^{-1} \tilde{R}_k^\top \tilde{\Upsilon}_k,$$

is an approximation of the on-line estimate $\sum_{j=0}^{\varsigma} C_j \Pi e_k^{-S\tau_j}$, and families of reduced order models at time $t_k$ can be easily defined.
Matching with prescribed eigenvalues

Determining at every $k$ the matrix $G_k$ such that

$$\sigma(F_k) = \{\lambda_{1,k}, \ldots, \lambda_{\nu,k}\}$$

for some prescribed values $\lambda_{i,k}$. The solution of this problem is well-known and consists in selecting $G_k$ such that

$$\sigma(S - G_k L) = \sigma(F_k).$$

This is possible for every $k$ and for all $\lambda_{i,k} \notin \sigma(S)$ and note that $G_k$ is independent from the estimate $\tilde{C}\Pi_k$. Note also that by observability of $(L, S)$, $G_k$ is unique at every $k$. 
These problems can be solved at each $k$ if and only if

$$\text{rank} \left[ \begin{array}{c} sI - S \\ \widetilde{C}\Pi_k \end{array} \right] = n,$$

for all $s \in \sigma(S)$ at $k$. Even though the asymptotic value of $\widetilde{C}\Pi_k$ satisfies this condition there is no guarantee that the condition holds for all $k$. However, if the condition holds for the asymptotic value, there exists $\bar{k} \gg 0$ such that for all $k \geq \bar{k}$ the equation has a solution.
Figure: Bode plot of the system (solid line), of the reduced order model at $t_k = 90s$ (dotted line), of the reduced order model at $t_k = 110s$ (dash-dotted line) and of the reduced order model at $t_k = 140s$ (dashed line). The circles indicate the interpolation points.
A nonlinear example

The averaged model of the DC-to-DC Ćuk converter is given by the equations

\[
\begin{align*}
L_1 \frac{d}{dt} i_1 &= -(1 - u)v_2 + E, \\
L_3 \frac{d}{dt} i_3 &= -uv_2 - v_4, \\
C_2 \frac{d}{dt} v_2 &= (1 - u)i_1 + ui_3, \\
C_4 \frac{d}{dt} v_4 &= i_3 - Gv_4, \\
y &= v_4,
\end{align*}
\]

\[
\text{Figure: } h(\pi(\omega)) = E \frac{\omega}{\omega - 1}
\]
A nonlinear example
Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research
Contents

» Introduction to moment matching
» The time domain approach to moment matching
» Model reduction for linear time-delay systems
» Model reduction for nonlinear time-delay systems
» Interpolation at infinitely many points
» Model reduction from input/output data
» A toolbox for the model reduction by moment matching
» Remarks and further research
A matlab toolbox for moment matching
A matlab toolbox for moment matching
Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research
Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
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Thank you for your attention!