Equiangular Tight Frames and the Restricted Isometry Property

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The Grey-Rankin Bound

Definition: A binary code $\{\mathbf{c}_n\}_{n=1}^{2N} \subseteq \mathbb{Z}_2^M$ is **self-complementary** if

$$\mathbf{c}_{n+N}(m) = \mathbf{c}_n(m) + 1 \pmod 2, \quad \forall m = 1, \ldots, M,\ n = 1, \ldots, N.$$

Its **distance** is the minimum Hamming distance between any two $\mathbf{c}_n$'s:

$$\Delta := \min_{n \neq n'} \mathrm{d}(\mathbf{c}_n, \mathbf{c}_{n'}).$$

In 1962, Grey proved self-complementary codes satisfy $2N \leq \frac{8\Delta(M-\Delta)}{M-(M-2\Delta)^2}$.

Example: $M = 6$, $N = 16$, $\Delta = 2$, $2N = 32 = \frac{8 \cdot 2 \cdot (6-2)}{6-(6-2\cdot 2)^2}$

$$\left[\begin{array}{cccccccccccccccc|cccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0\\
0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1\\
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1
\end{array}\right]$$

What does this have to do with compressed sensing?

Fifty Shades of Grey-Rankin Bound

"A Stimulating Analysis!" -Robert Calderbank

The Rankin Bound

A self-complementary code $\{\mathbf{c}_n\}_{n=1}^{2N} \subseteq \mathbb{Z}_2^M$ yields vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^{N} \subseteq \mathbb{R}^N$,

$$\boldsymbol{\varphi}_n(m) := \tfrac{1}{\sqrt{M}}(-1)^{\mathbf{c}_n(m)}.$$

Example: $M = 6$, $N = 16$

$$\boldsymbol{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The Grey-Rankin bound follows from a special case of a more general result: in 1956, Rankin showed that if there exists N antipodal pairs of spherical caps of radius $\frac{\theta}{2}$ in $\mathbb{R}^M$, then M, N and θ necessarily satisfy:

$$\sin^2\theta \leq \frac{(M-1)N}{M(N-1)}.$$

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$$\cos^2\theta \geq 1 - \frac{(M-1)N}{M(N-1)} = \frac{M(N-1)-(M-1)N}{M(N-1)} = \frac{N-M}{M(N-1)}.$$

Welch's (Rankin's) Lower Bound on Coherence

Theorem: [Welch 74] If $\{\varphi_n\}_{n=1}^N$ are unit norm vectors in $\mathbb{C}^M$, then

$$\sqrt{\frac{N-M}{M(N-1)}} \leq \mu := \max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle|$$

with equality $\Leftrightarrow$ $\{\varphi_n\}_{n=1}^N$ is an **equiangular tight frame (ETF)**, i.e.

- the rows of $\mathbf{\Phi}$ are orthogonal and have constant norm,
- the columns of $\mathbf{\Phi}$ are unit norm,
- the dot products of distinct columns of $\mathbf{\Phi}$ have constant magnitude.

Proof:

$$\begin{aligned}
0 &\leq \|\mathbf{\Phi}\mathbf{\Phi}^* - \tfrac{N}{M}\mathbf{I}\|_{\mathrm{Fro}}^2 \\
&= \|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}}^2 - \tfrac{N(N-M)}{M} \\
&= \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n' \neq n}}^{N} |\langle \varphi_n, \varphi_{n'} \rangle|^2 - \tfrac{N(N-M)}{M} \\
&\leq N(N-1) \max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle|^2 - \tfrac{N(N-M)}{M}.
\end{aligned}$$

Gray-Rankin Bound Equality $\Leftrightarrow$ ±1-Valued ETF

Theorem: [Grey 62; Jasper, Mixon & F 13] A self-complementary code $\{\mathbf{c}_n\}_{n=1}^{2N} \subseteq \mathbb{Z}_2^M$ achieves the Grey-Rankin bound if and only if $\{\boldsymbol{\varphi}_n\}_{n=1}^{N} \subseteq \mathbb{R}^M$, $\boldsymbol{\varphi}_n(m) := M^{-\frac{1}{2}}(-1)^{\mathbf{c}_n(m)}$ is an ETF.

Example:

$$\boldsymbol{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \frac{1}{6} \begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}$$

$$\boldsymbol{\Phi}^*\boldsymbol{\Phi} = \frac{1}{6} \begin{bmatrix} 6 & -2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 \\ -2 & 6 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 \\ 2 & 2 & 6 & -2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 \\ 2 & 2 & -2 & 6 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 \\ 2 & -2 & 2 & -2 & 6 & -2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 2 & -2 & 2 & -2 & 6 & 2 & 2 & -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 & 2 & 2 & 6 & -2 & -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\ -2 & 2 & -2 & 2 & 2 & 2 & -2 & 6 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 & 6 & -2 & 2 & 2 & 2 & -2 & 2 & -2 \\ 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 & -2 & 6 & 2 & 2 & -2 & 2 & -2 & 2 \\ 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 & 2 & 2 & 6 & -2 & 2 & -2 & 2 & -2 \\ 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & -2 & 6 & -2 & 2 & -2 & 2 \\ 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 6 & -2 & 2 & 2 \\ -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & -2 & 6 & 2 & 2 \\ -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 6 & -2 \\ 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & -2 & 6 \end{bmatrix}$$

Why Care?

- The previous result identifies certain optimal codes with ETFs.
- ETFs have optimally small coherence.
- Small coherence is important in compressed sensing.
- Coding theory, ETFs and compressed sensing have rich literatures.
- Can coding theory yield new results in compressed sensing or ETFs?
- Can compressed sensing or ETFs yield new results in coding theory?

ETF Construction Method 1: Difference Sets

Theorem: [Turyn 65; Strohmer & Heath 03; Xia, Zhou & Giannakis 05; Ding & Feng 07] If $\mathcal{D}$ is a subset of a finite abelian group $\mathcal{G}$, then the rows of the character table of $\mathcal{G}$ that correspond to $\mathcal{D}$ form an ETF if and only if $\mathcal{D}$ is a **difference set** in $\mathcal{G}$, namely if

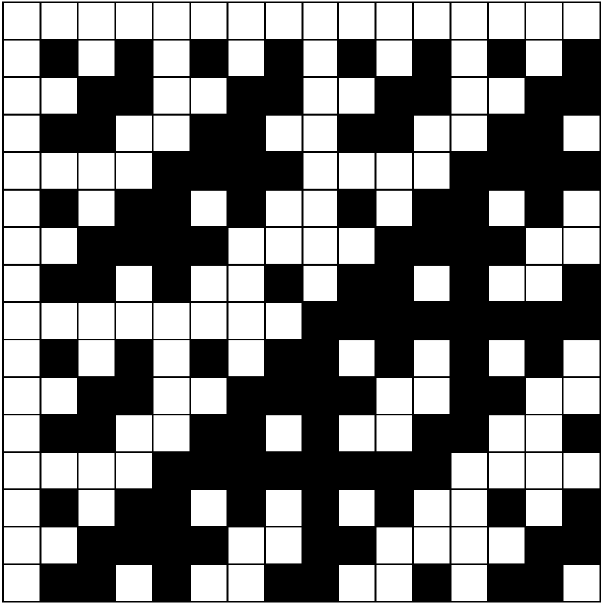
$$\lambda(g) := \#\{(d, d') \in \mathcal{D} \times \mathcal{D} : g = d - d'\}$$

is a constant function of $g \in \mathcal{G}$.

Example: $\mathcal{G} = \mathbb{Z}_2^4$, $M = \#(\mathcal{D}) = 6$, $N = \#(\mathcal{G}) = 16$, $\lambda = 2$:

$-$	(0, 0, 0, 0)	(0, 0, 1, 0)	(1, 0, 0, 0)	(1, 0, 0, 1)	(1, 1, 0, 0)	(1, 1, 1, 1)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 1, 0)	(1, 0, 0, 0)	(1, 0, 0, 1)	(1, 1, 0, 0)	(1, 1, 1, 1)
(0, 0, 1, 0)	(0, 0, 1, 0)	(0, 0, 0, 0)	(1, 0, 1, 0)	(1, 0, 1, 1)	(1, 1, 1, 0)	(1, 1, 0, 1)
(1, 0, 0, 0)	(1, 0, 0, 0)	(1, 0, 1, 0)	(0, 0, 0, 0)	(0, 0, 0, 1)	(0, 1, 0, 0)	(0, 1, 1, 1)
(1, 0, 0, 1)	(1, 0, 0, 1)	(1, 0, 1, 1)	(0, 0, 0, 1)	(0, 0, 0, 0)	(0, 1, 0, 1)	(0, 1, 1, 0)
(1, 1, 0, 0)	(1, 1, 0, 0)	(1, 1, 1, 0)	(0, 1, 0, 0)	(0, 1, 0, 1)	(0, 0, 0, 0)	(0, 0, 1, 1)
(1, 1, 1, 1)	(1, 1, 1, 1)	(1, 1, 0, 1)	(0, 1, 1, 1)	(0, 1, 1, 0)	(0, 0, 1, 1)	(0, 0, 0, 0)

Difference Set ETF Example with $M = 6$, $N = 16$



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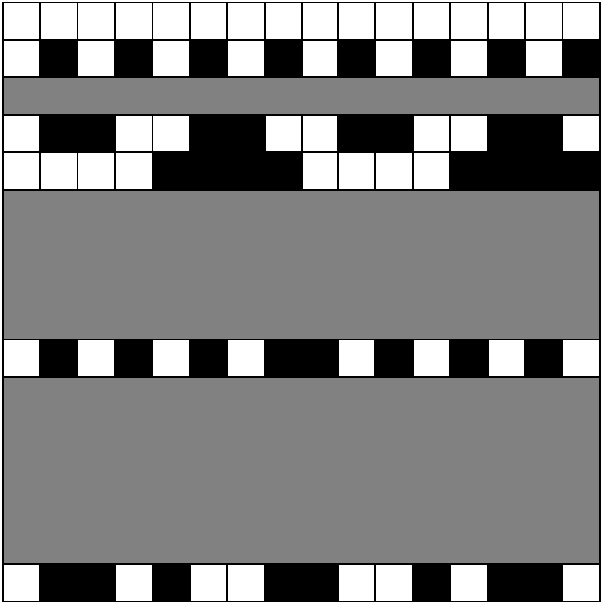

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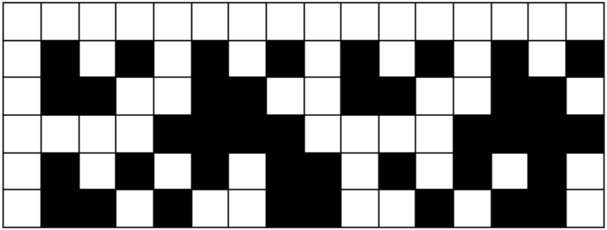

ETF $\Rightarrow$ RIP?

Example: $\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}.$

Recall: This ETF yields a 6×32 GRBE self-complementary code.

Question: Is this ETF a good RIP matrix, i.e., for what values $K \leq N$ and $\delta < 1$ do we have $\|\mathbf{\Phi}_{\mathcal{K}}^*\mathbf{\Phi}_{\mathcal{K}} - \mathbf{I}\|_2 \leq \delta$, $\forall \mathcal{K} \subseteq \{1, \dots, N\}$, $|\mathcal{K}| = K$?

The case for:

- Gershgorin circles: $\|\mathbf{\Phi}_{\mathcal{K}}^*\mathbf{\Phi}_{\mathcal{K}} - \mathbf{I}\|_2 \leq (K-1) \max\limits_{n \neq n'} |\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle|$.
- Random rows of a Hadamard matrix yields good RIP matrices [Rudelson & Vershynin 08]; our rows our "pseudo-random."

The case against:

- Square-root bottlebeck: by the Welch bound, Gershgorin circles only guarantee $\mathbf{\Phi}$ can be (K, δ)-RIP for some $\delta < 1$ for $K \sim \mathcal{O}(\sqrt{M})$.
- Difference sets yield **linear** GRBE codes (the codewords form a subspace of $\mathbb{Z}_2^M$); coding theory tells us such ETFs have $N \approx 2M$.

ETF Construction Method 2: Block Designs

Definition: A **(2,k,v)-Steiner system** is a set of v points $\mathcal{V}$ along with a collection $\mathcal{B}$ of b subsets (blocks) of $\mathcal{V}$ such that:

- every block contains exactly k points,
- every point is contained in exactly r blocks,
- any two distinct points are contained in exactly one block.

A Steiner system is **resolvable (Kirkman)** if its blocks $\mathcal{B}$ can be partitioned into subcollections so that each forms partition for $\mathcal{V}$.

Example: For when $k = 2$, $v = 4$, $r = 3$, $b = 6$ consider the $b \times v = 6 \times 4$ incidence matrix of all 2-subsets of $\{1, 2, 3, 4\}$:

$$\mathbf{B} = \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1, \ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

"$+$" $= 1$, "$-$" $= -1$

$$
\begin{bmatrix}
+ & + & 0 & 0 \\
0 & 0 & + & + \\
+ & 0 & + & 0 \\
0 & + & 0 & + \\
+ & 0 & 0 & + \\
0 & + & + & 0
\end{bmatrix}
\text{“}\otimes\text{”}
\begin{bmatrix}
+ & - & + & - \\
+ & + & - & - \\
+ & - & - & +
\end{bmatrix}
$$

$$
= \left[\begin{array}{cccc|cccc|cccc|cccc}
+ & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\
+ & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\
+ & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\
0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0
\end{array}\right]
$$

Example: Steiner ETF Construction

$$\text{“}+\text{''} = 1,\ \text{“}-\text{''} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{''} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\text{``}+\text{''} = 1,\ \text{``}-\text{''} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{``}\otimes\text{''} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\text{“}+\text{”} = 1, \ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

"$+$" $= 1$, "$-$" $= -1$

$$
\begin{bmatrix}
+ & + & 0 & 0 \\
0 & 0 & + & + \\
+ & 0 & + & 0 \\
0 & + & 0 & + \\
+ & 0 & 0 & + \\
0 & + & + & 0
\end{bmatrix}
\text{"}\otimes\text{"}
\begin{bmatrix}
+ & - & + & - \\
+ & + & - & - \\
+ & - & - & +
\end{bmatrix}
$$

$$
= \left[\begin{array}{cccc|cccc|cccc|cccc}
+ & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\
+ & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\
+ & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\
0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0
\end{array}\right]
$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1, \ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{"}+\text{"} = 1,\ \text{"}-\text{"} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{"}\otimes\text{"} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{"}+\text{"} = 1, \text{"}-\text{"} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{"}\otimes\text{"} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction ($\rightarrow$ Kirkman)

$$\text{“}+\text{''} = 1,\ \text{“}-\text{''} = -1$$

$$\left[\begin{array}{cccc} + & + & 0 & 0 \\ 0 & 0 & + & + \\ \hline + & 0 & + & 0 \\ 0 & + & 0 & + \\ \hline + & 0 & 0 & + \\ 0 & + & + & 0 \end{array}\right] \text{“}\otimes\text{''} \left[\begin{array}{cccc} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ \hline + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ \hline + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Kirkman ETF Construction

$$
\left[\begin{array}{cc|cc|cc}
+ & + & 0 & 0 & 0 & 0 \\
+ & - & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & + & + & 0 & 0 \\
0 & 0 & + & - & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & + & + \\
0 & 0 & 0 & 0 & + & -
\end{array}\right]
$$

$$
\times \left[\begin{array}{cccc|cccc|cccc|cccc}
+ & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\
\hline
+ & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\
\hline
+ & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\
0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
= \left[\begin{array}{cccccccccccccccc}
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{array}\right]
$$

Steiner/Kirkman ETFs and the Square-Root Bottleneck

Theorem: [Goethals & Seidel 70; F, Mixon & Tremain 12]
Any $(2, k, v)$-Steiner system generates an ETF with

$$M = b = \frac{v(v-1)}{k(k-1)}, \quad N = v(r+1) = v\left(\frac{v-1}{k-1} + 1\right).$$

The redundancy of this ETF is $\frac{N}{M} = k(1 + \frac{1}{r}) \approx k$.

Theorem: [Jasper, Mixon & F 13] If a Steiner system is resolvable its ETF can be rotated into a constant-amplitude **Kirkman ETF**.

Moreover, if $\{\varphi_n\}_{n=1}^N$ is any Steiner or Kirkman ETF for $\mathbb{C}^M$, then it is (K, δ)-RIP for some $\delta < 1$ if and only if

$$K \le \left(\frac{\rho M - 1}{\rho - 1}\right)^{\frac{1}{2}},$$

where $\rho = \frac{N}{M}$. Furthermore, every harmonic ETF generated via a McFarland difference set is one of these.

In particular, none of these ETFs can surpass the square-root bottleneck.

Our 6×16 Example Revisited

$$\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Question: Is this $\mathbf{\Phi}$ a good RIP matrix?

Answer: Surprisingly, no!
It was generated with a McFarland difference set in $\mathbb{Z}_2^4$.
As such, it can be rotated to a 6×16 Steiner ETF.
This means four of its columns are linearly dependent.
Thus, $\mathbf{\Phi}$ is not RIP with $K = 4$.

Open Problem: Can any ETFs be good RIP matrices? In particular, does the symmetry required to be as "pairwise nearly-orthonormal" prevent them from having higher orders of "near orthonormality"?

Take Away

- GRBE codes are equivalent to ± 1-valued ETFs.
- The (Rankin-)Welch bound predates Welch's work by almost two decades.
- All known constructions of ± 1-valued ETFs (as well as most constructions of ETFs in general) are provably incapable of breaking compressed sensing's square-root bottleneck (low spark).
- Coding theory informs ETFs: every ± 1-valued ETF arising from a difference set must have $N \approx 2M$.
- ETFs inform coding theory: assuming the Hadamard conjecture is true and using asymptotic results on the existence of resolvable block designs, there exists infinite families of ± 1-valued ETFs for every redundancy approximately equal to 2 mod 4.
- Moving forward, what does compressed sensing (random matrix theory) tell us about coding theory?

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