Reduced order modeling of parameter dependent nonlinear eigenvalue bifurcation problems

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Mathematics for key technologies

Sparsity in PDE solutions

- Numerical solution of PDE $Lu = f$, with differential operator L in a domain $\Omega \subset \mathbb{R}^d$ with boundary Γ and appropriate boundary conditions given on Γ.
- Let $\mathcal{V}$ be an ansatz function space in which we know or expect the solution to be.
- Choose another (or the same) space $\mathcal{W}$ as test space.
- Classical Galerkin or Petrov-Galerkin approach: Seek solution u in some finite dimensional ansatz space $\mathcal{V}_n \subset \mathcal{V}$ (spanned by a basis or frame) $\mathcal{B} = \{\phi_1, \dots, \phi_n\}$, i.e. the solution is represented as $u = \sum_{i=1}^n u_i \phi_i$ and $(Lu - f, w) = 0$ of $|(Lu - f, w)| < \epsilon$ for all $w \in W$, where ϵ is a given tolerance.

Questions

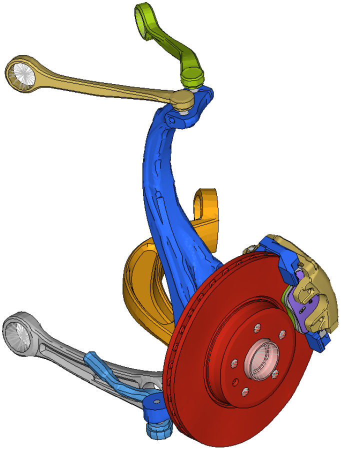
- Can u be sparsely represented in $\mathcal{V}$? Sure if the solution lies in $\mathcal{V}$, just take $u \in \mathcal{B}$.
- Can u be sparsely represented in $\mathcal{V}_n \subset \mathcal{V}$.
- What is a good basis/frame of $\mathcal{V}_n$ so that u can be sparsely represented.
- What are conditions for the basis/frame so that the finite dimensional version $L_n u_n = f_n$ has a sparse L_h, or a sparse inverse L_h^{-1}.
- Is there a cheap ($O(n)$) method to get a sparse solution?
- Can we have all this together?
- Is there a 'eierlegende Wollmilchsau', a swiss army knife for PDE solution?

Outline

Current project with Audi and Opel and several SMEs (2012-14)
Joint with N. Gräbner, U. von Wagner, TU Berlin, Mechanics and N. Hoffmann, TU Hamburg-Harburg, Mechanics,
S. Quraishi, C. Schröder, TU Berlin Mathematics.

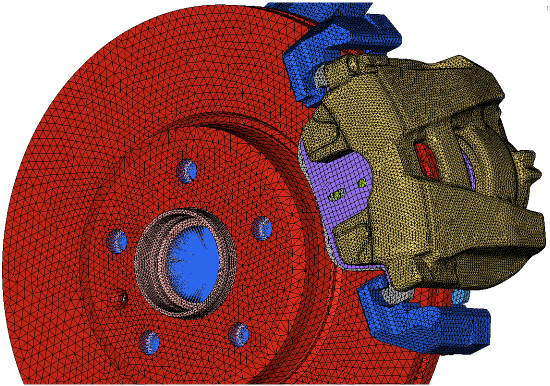
Goals:

- Develop mechanics based discrete FE model of disk brake.
- Identification of energy dissipation effects.
- Model and simulate nonlinear effects in brake squeaking.
- Reduced order (compressed) model for a given range of disk speeds.
- Sparse representation of operator and solution.
- Finally, passive and active remedies to avoid squeaking.



View of the brake model

Finite element model



Differential-algebraic equation (DAE)

$$M\ddot{q} + (C_1 + \frac{\omega_{ref}}{\omega} C_r + \frac{\omega}{\omega_{ref}} C_g)\dot{q} + (K_1 + K_r + (\frac{\omega}{\omega_{ref}})^2 K_g)q = f.$$

- M is symmetric semi-definite mass matrix,
- $C = C_1 + C_g + C_r$ is a 'damping matrix'
  - C_1 is symmetric,
  - C_g (due to gyroscopic effects) is skew-symmetric,
  - C_r is friction induced damping (symmetric),
  - ω is the angular velocity, ω_{ref} reference.
- $K = K_1 + K_r + K_g$ is a 'stiffness matrix'
  - K_1 is symmetric, dominant part,
  - K_r describes circulatory effects (non symmetric),
  - K_g is geometric stiffness matrix.

Complex eigenvalue analysis

- Setting $q(t) = e^{\lambda t} u$, we get a quadratic eigenvalue problem (QEP):

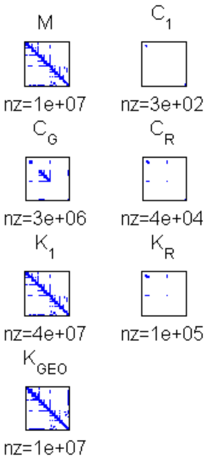
$$P_\omega(\lambda)u = (\lambda^2 M + \lambda C(\omega) + K(\omega))u = 0.$$

- Likelihood of brake to squeal is correlated with magnitude of positive real part of eigenvalue.
- Compute eigenvalues in right half plane for lots of parameter values e.g. $\omega \in (2\pi,\ 2\pi \times 20)$.

Nature of FE matrices

$$C = C_1 + \frac{\omega_{ref}}{\omega} C_r + \frac{\omega}{\omega_{ref}} C_g,$$
$$K = K_1 + K_r + \left(\frac{\omega}{\omega_{ref}}\right)^2 K_g$$
$$n = 842,638,\ \omega_{ref} = 5, \omega = 17 \times 2\pi$$

matrix	**pattern**	**2-norm**	**structural rank**
M	symm	5e-2	842,623
D_1	symm	1e-19	160
$\mathbf{D_G}$	**skew**	**1.5e-1**	**217500**
$\mathbf{D_R}$	**symm**	**7e-2**	**2120**
$\mathbf{K_1}$	**symm**	**2e13**	**full**
K_R	-	3e4	2110
K_{GEO}	symm	40	842,623



Challenges

- The discrete modeling is done directly with matrices, so space discretization cannot easily be done in an adaptive FEM way.
- The set of ansatz functions (dictionary) is fixed, not a choice.
- It is difficult to enrich the space with 'better functions'.
- We have to work in an algebraic framework.
- How to bring in sparsity?

Outline

Projection approach

- Project QEP: $P_\omega(\lambda)x = (\lambda^2 M + \lambda C(\omega) + K(\omega))x = 0$ into a small d-dimension subspace Q independent of ω.
- Projected QEP
  - $\tilde{P}_\omega(\lambda) = Q^T P_\omega(\lambda) Q = (\lambda^2 Q^T M Q + \lambda Q^T C(\omega) Q + Q^T K(\omega) Q)$
- How to choose Q to get sufficiently accurate approximation of eigenvalues with positive real part;
- How to choose Q to capture the important (to analyze and modify the squeaking) dynamics of the system;
- Ideally Q should contain good approximations to the wanted eigenvectors for all parameter values;
- We should be able to construct Q in a reasonable amount of time.

1. Traditional approach, often with Algebraic Multi Level Sub-structuring (AMLS).
2. New proper orthogonal decomposition (POD) based approach.

- Traditional approach to get a subspace Q:
  - Q_{TRAD} matrix of dominant eigenvectors.
  - Select dominant eigenvectors by solving the GEVP $K_1 v = -\lambda^2 M v$
- Advantages:
  - Only need to solve a large sparse, symmetric and definite GEVP.
- Disadvantages:
  - Subspace does not take into account damping and parameter dependence.
  - The reduced model often does not capture the important dynamics.
  - Poor approximation of true eigenpairs.

Undamped model without circulatory and gyroscopic forces:
$(\lambda^2 M + K + K_g)x = 0.$

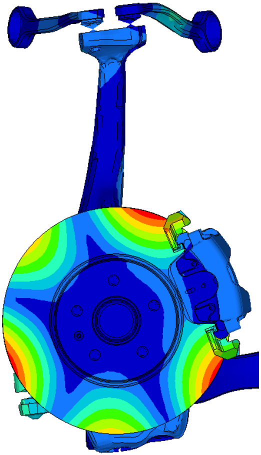


Figure: Trad. eigenmode at 1859 Hz

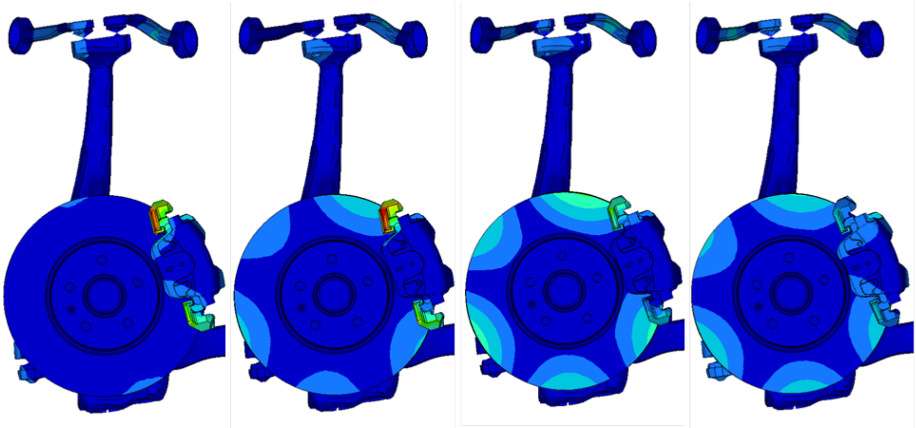
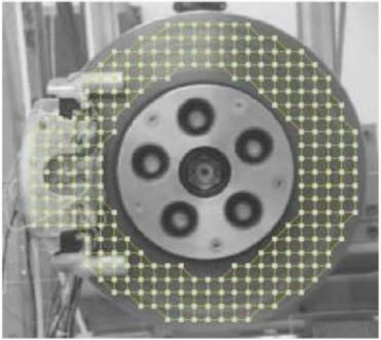
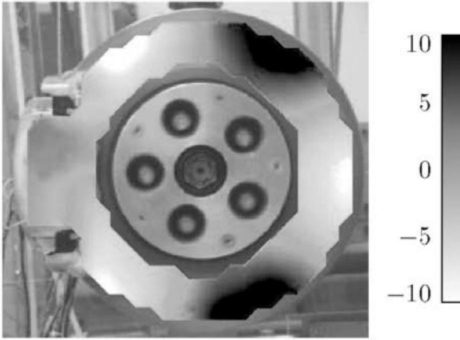


Figure: Eigenform at 1873 Hz with positive real part and a phase of 0, 45, 90, and 135.

Measurement of brake vibrations



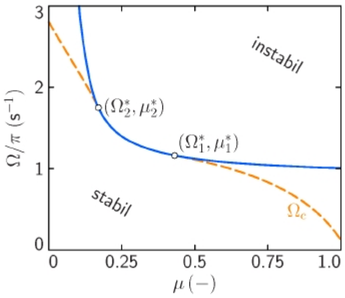
Gitter der Messpunkte



Betriebsschwingform (1750 Hz)

Measurements indicate subcritical Hopf bifurcations, i.e. eigenvalues crossing imaginary axis for certain disk frequencies. Traditional approach deals only with purely imaginary evs.

Stability regions, linear vs. nonlinear



Bifurcation diagram linear analysis (blue), nonlinear analysis (red). Coefficient of friction μ via disk frequency Ω.

Outline

We use the classical companion linearization to turn the quadratic into a linear generalized eigenvalue problem

$$A_\tau(\omega)x(\omega) = \lambda_\tau B_\tau(\omega)x(\omega)$$

with

$$\begin{bmatrix} K_\tau(\omega) & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} x(\omega) \\ \lambda_\tau(\omega)x(\omega) \end{bmatrix} = \lambda_\tau(\omega) \begin{bmatrix} -C_\tau(\omega) & -M_\tau \\ I_n & 0 \end{bmatrix} \begin{bmatrix} x(\omega) \\ \lambda_\tau x(\omega) \end{bmatrix}.$$

Shift and invert Arnoldi

- Compute eigenvalue and eigenvector approximations near a given shift point τ via the Shift-and-invert Arnoldi method.
- Given $v_0 \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$, the *Krylov subspace* of $\mathbb{C}^n$ of order k associated with A is

$$\mathcal{K}_k(A, v_0) = span\{v_0, Av_0, A^2 v_0 ..., A^{k-1} v_0\}.$$

- Arnoldi obtains an orthonormal basis of this space and an Arnoldi relation

$$AV_k = V_k H_k + fe_k^*,$$

- The columns of V_k are approximation of k-dimensional invariant subspace of A.
- H_k is upper Hessenberg, its evs are Ritz approximations to evs of A associated to V_k.
- We apply Arnoldi with shift τ and frequency ω to the matrix $A = B_\tau(\omega)^{-1} A_\tau(\omega)$. In every step we have to multiply with $A_\tau(\omega)$ and to solve a linear system with the matrix $B_\tau(\omega)$.

SVD projection

- We construct a *measurement matrix* $X \in \mathbb{R}^{n,km}$ containing the 'unstable' eigenvectors for a sequence of angular velocities,

$$X = [X(\omega_1), X(\omega_2), X(\omega_3), \ldots X(\omega_k)]$$

- Perform a singular value decomposition (SVD) of X

$$X = [u_1, u_2, u_3, \ldots u_{km}] \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_{km} \end{bmatrix} [v_1, v_2, v_3, \ldots v_{km}]^T$$

Compression

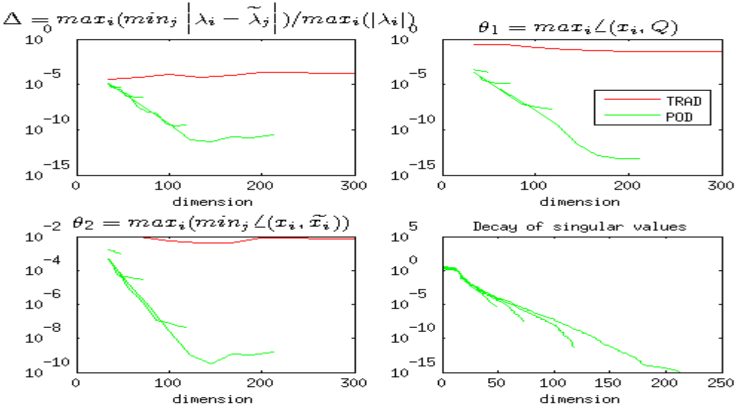
- We use approximation

$$X \approx [u_1, u_2, u_3, \ldots u_d] \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_d \end{bmatrix} [v_1, v_2, v_3, \ldots v_d]^T$$

  provided $\sigma_{d+1}, \sigma_{d+2}, \ldots \sigma_{km}$ are small.
- We choose $Q = [u_1, u_2, u_3, \ldots u_d]$ to project $P_\omega(\lambda)$.

Some results on small $n \approx 5000$ matrices

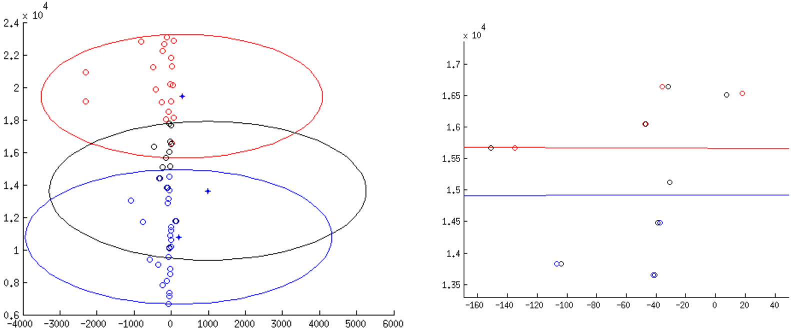
- POD for uniformly spaced p parameters
  $p = 2^j + 1, j = 0, 1, 2, 3$



- Increasing dimension does not improve TRAD approach

Realistic $n \approx 800,000$ matrices

- The evp is completely singular M, D have a 12 dimensional common nullspace and K has relative size 10^{-14} in that nullspace.
- Shifted matrix $\widetilde{K_\tau} = \tau^2 \widetilde{M} + \tau \widetilde{C} + \widetilde{K}$ has condition number $\sim 10^{14}$ for a range of target points. Most likely due to bad FEM model.
- Need to solve many large scale evps to get measurement matrix $\widetilde{X} = [X(\omega_1),\ X(\omega_2),\ X(\omega_3) \cdots\ X(\omega_p)]$.
- It is not clear which parameter values ω_i are important.
- Where to look for eigenvalues in the right half plane.
- Scaling of matrices with scalar parameters to make them comparable in norm.
- Diagonal scaling of matrices to improve conditioning.

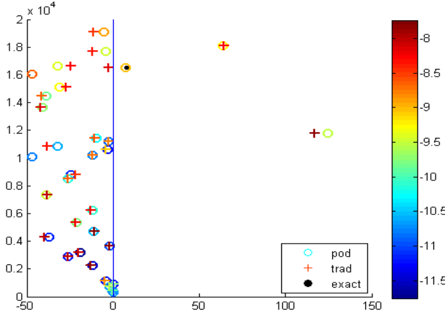


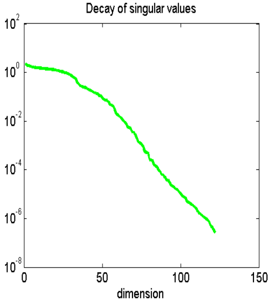
- Different shift gives different evs in the overlapping region.

- Construction of subspace (One time investment)
  - Each shift of eigs (Arnoldi method) $\sim$ 20 min
  - Eigenpairs for each parameter value $\sim$ 3 targets $\sim$ 1 hour
  - POD measurement vectors for 2 parameters $\sim$ 2 hours (or just 20 min on 6 processors)
  - Constructing POD subspace (SVD) $\sim$ 1 min
  - Constructing 300 dimensional TRAD subspace $\sim$ 45 min

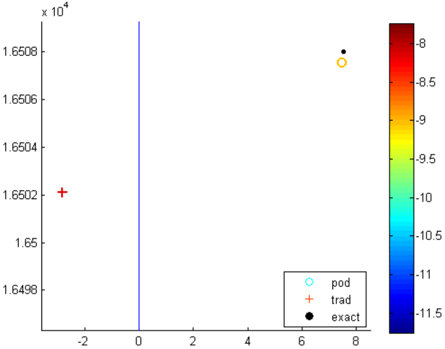
- Solution for every ω
  - Solution with 300 dimensional TRAD subspace $\sim$ 30 sec
  - Solution with 100 dimensional POD subspace $\sim$ 10 sec

- POD model for $\omega = [1, 20] \times 2\pi$
- Color coded with residual $\mathcal{R} = \frac{\|(\lambda_i^2 M + \lambda_i C + K)u_i\|_\infty}{\|(|\lambda_i|^2|M| + |\lambda_i||C| + |K|)|u_i|\|_\infty}$
- U_{POD}: 100, U_{TRAD}: 300 (Industry Recommendation)

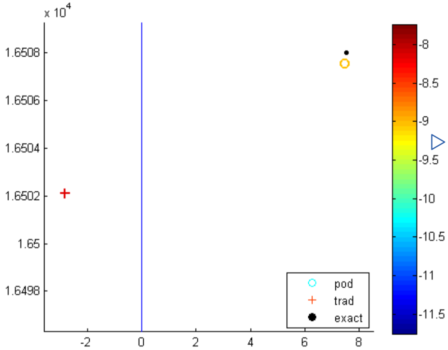




- all +'s are red (TRAD approach has very high residual)

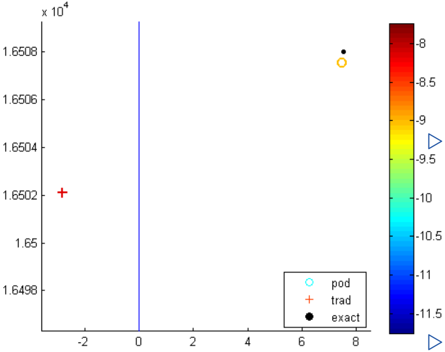


- Place shift point $\tau = 7.5 + 16500i$ near an eigenvalue found from POD



- Place shift point $\tau = 7.5 + 16500i$ near an eigenvalue found from POD
- Running eigs with this shift result in an exact eigenvalue $\lambda = 7.5414 + 16508i$ very close to POD result

TRAD misses important eigenvalue



- Place shift point $\tau = 7.5 + 16500i$ near an eigenvalue found from POD
- Running eigs with this shift result in an exact eigenvalue $\lambda = 7.5414 + 16508i$ very close to POD result
- TRAD misses it

What did we learn?

- POD is better than traditional approach but not satisfactory.
- Discrete FE and quasi-uniform grids followed by expensive model reduction is really a waste.
- Numerical linear algebra methods that we currently use are not efficient (in particular those in commercially available codes).
- For evp everything is partially heuristic.
- Can we get error estimates? Can we bring in adaptivity? Dictionary learning?
- Can we disprove the engineers that say that uniform mesh and brute force linear algebra is best.

A compressed sensing point of view

- The vectors $q(t)$ represent coefficient vectors for the infinite dimensional solution represented in an FEM basis $\phi_1(x,t), \dots, \phi_N(x,t)$ in space-time.
- The eigenvectors $x_i(\omega)$ also represent coefficient vectors in this FEM basis to synthesize the fundamental solution matrix of the DAE.
- Every eigenvector $x_i(\omega)$ is the coefficient vector of a non-sparse function $\xi_i(x,\omega,t)$, because it typically linearly combines many FEM basis functions.
- The POD basis represents a small set of linear combinations of the $\xi_i(\omega)$, given by functions $\psi_j(x,t)$ $j = 1, \dots, d$ which are independent of ω.
- Consider the dictionary $\mathcal{D} = \{\phi_1(x,\omega,t), \dots, \phi_N(x,\omega,t)\} \cup \{\xi_1(x,\omega,t), \dots, \xi_\ell(x,\omega,t)\}$.
- Choosing the POD basis is selecting a small 'sparse' set of linear combinations from $\mathcal{D}$.

Conclusions and Questions.

- Real world industrial problems as motivation for studying, functions spaces, dictionaries, . . .
- Can we use this analogy to get convergence proofs, error bounds, complexity analysis?
- What kind of sparsity should we go for?
- How should we construct FE dictionaries?
- Can we convince the engineers?
- Can we make this practical?
- Can we remove the brake squeal?

Thank you very much
for your attention.