Interpolation via weighted ℓ_1-minimization

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Joint work with Rachel Ward (University of Texas at Austin)

Function interpolation

Aim

Given a function $f : \mathcal{D} \to \mathbb{C}$ on a domain $\mathcal{D}$ reconstruct or interpolate f from sample values $f(t_1), \ldots, f(t_m)$.

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- (Linear) polynomial interpolation
  - assumes (classical) smoothness in order to achieve error rates
  - works with special interpolation points (e.g. Chebyshev points).
- Compressive sensing
  - reconstruction nonlinear
  - assumes sparsity (or compressibility) of a series expansion in terms of a certain basis (e.g. trigonometric bases)
  - fewer (random!) sampling points than degrees of freedom

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This talk: Combine sparsity and smoothness!

A classical interpolation result

$C^r([0,1]^d)$: r-times continuously differentiable periodic functions

Existence of set of sampling points $t_1, \ldots, t_m$ and linear reconstruction operator $R : \mathbb{C}^m \to C^r([0,1]^d)$ such that for every $f \in C^r([0,1]^d)$ the approximation $\widetilde{f} = R(f(t_1), \ldots, f(t_m))$ satisfies the optimal error bound

$$\|f - \widetilde{f}\|_\infty \leq C m^{-r/d} \|f\|_{C^r}.$$

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Curse of dimension: Need about $m \geq C_f \varepsilon^{-d/r}$ samples for achieving error $\varepsilon < 1$.

Exponential scaling in d cannot be avoided using only smoothness (DeVore, Howard, Micchelli 1989 – Novak, Wozniakowski 2009).

Sparse representation of functions

$\mathcal{D}$: domain endowed with a probability measure ν
$\psi_j : \mathcal{D} \to \mathbb{C}$, $j \in \Gamma$ (finite or infinite)
$\{\psi_j\}_{j\in\Gamma}$ orthonormal system:

$$\int_{\mathcal{D}} \psi_j(t)\overline{\psi_k(t)}d\nu(t) = \delta_{j,k}, \quad j,k \in \Gamma$$

We consider functions of the form

$$f(t) = \sum_{j\in\Gamma} x_j \psi_j(t)$$

f is called s-sparse if $\|\mathbf{x}\|_0 := \{\ell : x_\ell \neq 0\} \leq s$ and compressible if the error of best s-term approximation error

$$\sigma_s(f)_q := \sigma_s(\mathbf{x})_q := \inf_{\mathbf{z}:\|\mathbf{z}\|_0\leq s} \|\mathbf{x}-\mathbf{z}\|_q \quad (0 < q \leq \infty)$$

is small.

Aim: Reconstruction of sparse/compressible f from samples!

Fourier Algebra and Compressibility

Fourier algebra $A_p = \{f \in C[0,1] : \|\!| f |\!\|_p < \infty\}$, $0 < p \leq 1$,

$$\|\!| f |\!\|_p := \|\mathbf{x}\|_p = (\sum_{j\in\mathbb{Z}} |x_j|^p)^{1/p}, \quad f(t) = \sum_{j\in\mathbb{Z}} x_j \psi_j(t).$$

Motivating example trigonometric system:
$\mathcal{D} = [0,1]$, ν Lebesgue measure,

$$\psi_j(t) = e^{2\pi i j t}, \quad t \in [0,1], j \in \mathbb{Z}.$$

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Compressibility via Stechkin estimate

$$\sigma_s(f)_q = \sigma_s(\mathbf{x})_q \leq s^{1/q-1/p} \|\mathbf{x}\|_p = s^{1/q-1/p} \|\!| f |\!\|_p, \quad p < q.$$

Since $\|f\|_\infty := \sup_{x\in[0,1]} |f(t)| \leq \|\!| f |\!\|_1$, the best s-term approximation $f_0 = \sum_{j\in S} x_j \psi_j$, $|S| = s$, satisfies

$$\|f - f_0\|_\infty \leq s^{1-1/p} \|\!| f |\!\|_p, \quad p < 1.$$

Trigonometric system: smoothness and weights

$\psi_j(t) = e^{2\pi ijt}$, $j \in \mathbb{Z}$, $t \in [0,1]$
Derivatives satisfy $\|\psi_j'\|_\infty = 2\pi|j|$, $j \in \mathbb{Z}$.

For $f(t) = \sum_j x_j \psi_j(t)$ we have

$$\begin{aligned}
\|f\|_\infty + \|f'\|_\infty &= \|\sum_j x_j \psi_j\|_\infty + \|\sum_j x_j \psi_j'\|_\infty \\
&\le \sum_{j\in\mathbb{Z}} |x_j|(\|\psi_j\|_\infty + \|\psi_j'\|_\infty) \\
&= \sum_{j\in\mathbb{Z}} |x_j|(1 + 2\pi|j|) =: \|\mathbf{x}\|_{\omega,1}.
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\end{aligned}$$

Weights model smoothness!

Combine with sparsity (compressibility)
$\rightarrow$ weighted ℓ_p-spaces with $0 < p \leq 1$

Weighted norms and weighted sparsity

For a weight $\omega = (\omega_j)_{j\in\Gamma}$ with $\omega_j \geq 1$, introduce

$$\|\mathbf{x}\|_{\omega,p} := (\sum_{j\in\Gamma} |x_j|^p \omega_j^{2-p})^{1/p}, \quad 0 < p \leq 2.$$

Special cases:

$$\|\mathbf{x}\|_{\omega,1} = \sum_{j\in\Gamma} |x_j|\omega_j, \qquad \|\mathbf{x}\|_{\omega,2} = \|\mathbf{x}\|_2$$

Weighted sparsity

$$\|\mathbf{x}\|_{\omega,0} := \sum_{j:x_j\neq 0} \omega_j^2$$

$\mathbf{x}$ is called weighted s-sparse if $\|\mathbf{x}\|_{\omega,0} \leq s$.

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Note: If $\|\mathbf{x}\|_{\omega,0} \leq s$ then $\|\mathbf{x}\|_{\omega,1} \leq \sqrt{s}\|\mathbf{x}\|_{\omega,2}$.

Weighted best approximation

Weighted best s-term approximation error

$$\sigma_s(\mathbf{x})_{\omega,p} := \inf_{\mathbf{z}:\|\mathbf{z}\|_{\omega,0}\leq s} \|\mathbf{x}-\mathbf{z}\|_{\omega,p}$$

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Theorem (Weighted Stechkin estimate)

For a weight ω, a vector $\mathbf{x}$, $0 < p < q \leq 2$ and $s > \|\omega\|_\infty^2$,

$$\sigma_s(\mathbf{x})_{\omega,q} \leq (s - \|\omega\|_\infty^2)^{1/q-1/p}\|\mathbf{x}\|_{\omega,p}.$$

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If $s \ge 2\|\omega\|_\infty^2$, say, then

$$\sigma_s(\mathbf{x})_{\omega,q} \le C_{p,q} s^{1/q-1/p}\|\mathbf{x}\|_{\omega,p}, \quad C_{p,q} = 2^{1/p-1/q}.$$

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Lower bound on s natural because otherwise the single element set $S = \{j\}$ with $\omega_j = \|\omega\|_\infty$ not allowed as support set.

(Weighted) Compressive Sensing

Recover a weighted s-sparse (or weighted-compressible) vector $\mathbf{x}$ from measurements $\mathbf{y} = A\mathbf{x}$, where $A \in \mathbb{C}^{m \times N}$ with $m < N$.

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Weighted ℓ_1-minimization

$$\min_{\mathbf{z} \in \mathbb{C}^N} \|\mathbf{z}\|_{\omega,1} \quad \text{subject to } A\mathbf{z} = \mathbf{y}$$

"Noisy" version

$$\min_{\mathbf{z} \in \mathbb{C}^N} \|\mathbf{z}\|_{\omega,1} \quad \text{subject to } \|A\mathbf{z} - \mathbf{y}\|_2 \leq \eta$$

Weighted restricted isometry property (WRIP)

Definition

The weighted restricted isometry constant $\delta_{\omega,s}$ of a matrix $A \in \mathbb{C}^{m \times N}$ is defined to be the smallest constant such that

$$(1 - \delta_{\omega,s})\|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1 + \delta_{\omega,s})\|\mathbf{x}\|_2^2$$

for all $\mathbf{x} \in \mathbb{C}^N$ with $\|\mathbf{x}\|_{\omega,0} = \sum_{\ell : x_\ell \neq 0} \omega_j^2 \leq s$.

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Since $\omega_j \geq 1$ by assumption, the "classical" RIP implies the WRIP, $\delta_{\omega,s} \leq \delta_{\mathbf{1},s} = \delta_s$.

Alternative name:
Weighted Uniform Uncertainty Principle (WUUP)

Recovery via weighted ℓ_1-minimization

Theorem

Let $A \in \mathbb{C}^{m \times N}$ and $s \geq 2\|\omega\|_\infty^2$ such that $\delta_{\omega,3s} < 1/3$. For $\mathbf{x} \in \mathbb{C}^N$ and $\mathbf{y} = A\mathbf{x} + \mathbf{e}$ with $\|\mathbf{e}\|_2 \leq \eta$ let $\mathbf{x}^\sharp$ be a minimizer of

$$\min \|\mathbf{z}\|_{\omega,1} \quad \textit{subject to } \|A\mathbf{z} - \mathbf{y}\|_2 \leq \eta.$$

Then

$$\|\mathbf{x} - \mathbf{x}^\sharp\|_{\omega,1} \leq C_1 \sigma_s(\mathbf{x})_{\omega,1} + D_1 \sqrt{s}\eta,$$

$$\|\mathbf{x} - \mathbf{x}^\sharp\|_2 \leq C_2 \frac{\widetilde{\sigma}_s(\mathbf{x})_{\omega,1}}{\sqrt{s}} + D_2 \eta.$$

Function interpolation

$\{\psi_j\}_{j\in\Gamma}$, finite ONS on $\mathcal{D}$ with respect to probability measure ν.

Given samples $y_1 = f(t_1), \ldots, y_m = f(t_m)$ of $f(t) = \sum_{j\in\Gamma} x_j \psi_j(t)$ reconstruction amounts to solving

$$\mathbf{y} = A\mathbf{x}$$

with **sampling matrix** $A \in \mathbb{C}^{m\times N}$, $N = |\Gamma|$, given by

$$A_{\ell k} = \psi_k(t_\ell).$$

Use weighted ℓ_1-minimization to recover weighted-sparse or weighted-compressible $\mathbf{x}$ when $m < |\Gamma|$.

Choose $t_1, \ldots, t_m$ i.i.d. at random according to ν in order to analyze the WRIP of the sampling matrix.

Weighted RIP of random sampling matrix

$\psi_j : \mathcal{D} \to \mathbb{C}$, $j \in \Gamma$, $N = |\Gamma| < \infty$, ONS w.r.t. prob. measure ν.
Weight ω with $\|\psi_j\|_\infty \leq \omega_j$.

Sampling points $t_1, \ldots, t_m$ taken i.i.d. at random according to ν.
Random sampling matrix $A \in \mathbb{C}^{m \times N}$ with entries $A_{\ell j} = \psi_j(t_\ell)$.

Theorem (R, Ward '13)

If

$$m \geq C\delta^{-2} s \max\{\ln^3(s)\ln(N), \ln(\varepsilon^{-1})\}$$

then the weighted restricted isometry constant of $\frac{1}{\sqrt{m}}A$ *satisfies* $\delta_{\omega,s} \leq \delta$ *with probability at least* $1 - \varepsilon$.

Generalizes previous results (Candès, Tao – Rudelson, Vershynin – Rauhut) for systems with $\|\psi_j\|_\infty \leq K$ for all $j \in \Gamma$, where the sufficient condition is $m \geq C\delta^{-2}K^2 s \ln^3(s)\ln(N)$.

Abstract weighted function spaces

$$A_{\omega,p} = \{f : f(t) = \sum_{j\in\Gamma} x_j \psi_j(t),\ |||\, f \,|||_{\omega,p} := \|\mathbf{x}\|_{\omega,p} < \infty\}$$

If $\omega_j \geq \|\psi_j\|_\infty$ then

$$\|f\|_\infty \leq |||\, f \,|||_{\omega,1}.$$

If $\omega_j \geq \|\psi_j\|_\infty + \|\psi_j'\|_\infty$ (when $\mathcal{D} \subset \mathbb{R}$) then

$$\|f\|_\infty + \|f'\|_\infty \leq |||\, f \,|||_{\omega,1},$$

and so on...

Interpolation via weighted ℓ_1-minimization

Theorem

Assume $N = |\Gamma| < \infty$, $\omega_j \geq \|\psi_j\|_\infty$ *and* $0 < p < 1$. *Choose* $t_1, \ldots, t_m$ *i.i.d. at random according to* ν *where* $m \geq Cs \log^3(s) \log(N)$ *for* $s \geq 2\|\omega\|_\infty^2$. *Then with probability at least* $1 - N^{-\log^3(s)}$ *the following holds for each* $f \in A_{\omega,p}$. *Let* $\mathbf{x}^\sharp$ *be the solution of*

$$\min_{\mathbf{z} \in \mathbb{C}^N} \|\mathbf{z}\|_{\omega,1} \quad \textit{subject to} \sum_{j \in \Gamma} z_j \psi_j(t_\ell) = f(t_\ell), \ \ell = 1, \ldots, m$$

and set $f^\sharp(t) = \sum_{j \in \Gamma} x_j^\sharp \psi_j(t)$. *Then*

$$\|f - f^\sharp\|_\infty \leq |||f - f^\sharp|||_{\omega,1} \leq C_1 s^{1-1/p} |||f|||_{\omega,p},$$
$$\|f - f^\sharp\|_{L^2_\nu} \leq C_2 s^{1/2-1/p} |||f|||_{\omega,p}.$$

Error bound in terms of the number of samples

Solving $m \asymp s \log^3(s) \log(N)$ for s and inserting into error bounds yields

$$\|f - f^\sharp\|_\infty \leq |||f - f^\sharp|||_{\omega,1} \leq C_1 \left(\frac{m}{\log(N)^4}\right)^{1-1/p} |||f|||_{\omega,p},$$

$$\|f - f^\sharp\|_{L^2_\nu} \leq C_2 \left(\frac{m}{\log(N)^4}\right)^{1/2-1/p} |||f|||_{\omega,p}.$$

Quasi-interpolation in infinite-dimensional spaces

$|\Gamma| = \infty$, $\lim_{|j|\to\infty} \omega_j = \infty$ and $\omega_j \geq \|\psi_j\|_\infty$.

Theorem

Let $f \in A_{\omega,p}$ for some $0 < p < 1$, and set $\Gamma_s = \{j \in \Gamma : \omega_j^2 \leq s/2\}$ for some s. Choose $t_1, \ldots, t_m$ i.i.d. at random according to ν where $m \geq Cs \max\{\log^3(s)\log(|\Gamma_s|), \log(\varepsilon^{-1})\}$. With $\eta = |||\, f \,|||_{\omega,p}/\sqrt{s}$ let $\mathbf{x}^\sharp$ be the solution to

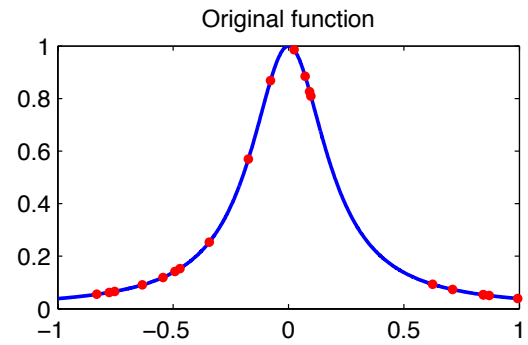
$$\min_{\mathbf{z}\in\mathbb{C}^{\Gamma_s}} \|\mathbf{z}\|_{\omega,1} \quad \textit{subject to } \|(f(t_\ell) - \sum_{j\in\Gamma_s} z_j\psi_j(t_\ell))_{\ell=1}^m\|_2 \leq \eta\sqrt{m}.$$

and put $f^\sharp(t) = \sum_{j\in\Gamma_s} x_j^\sharp \psi_j(t)$. Then with probability exceeding $1-\varepsilon$

$$\|f - f^\sharp\|_\infty \leq |||\, f - f^\sharp \,|||_{\omega,1} \leq C_1 s^{1-1/p} |||\, f \,|||_{\omega,p},$$
$$\|f - f^\sharp\|_{L^2_\nu} \leq C_2 s^{1/2-1/p} |||\, f \,|||_{\omega,p}.$$

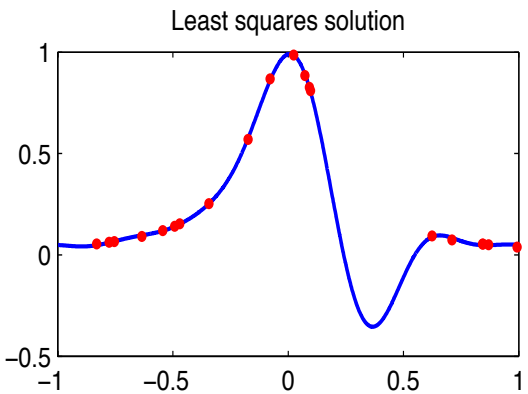
Ideally $|\Gamma_s| \leq Cs^\alpha$. Then $m \geq C_\alpha s \ln^4(s)$ samples are sufficient.

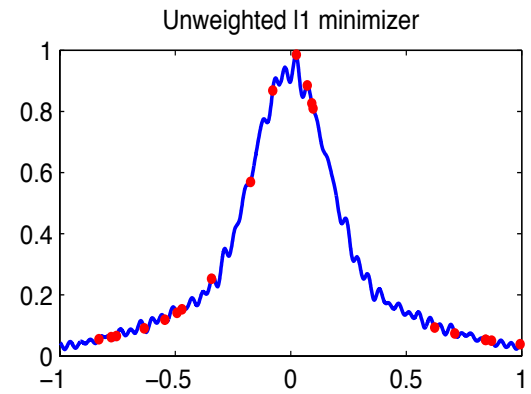
Numerical example I for the trigonometric system

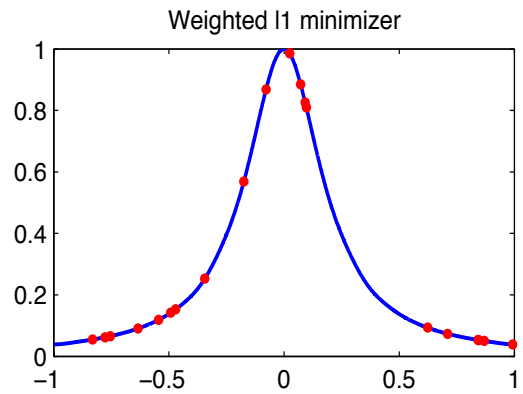


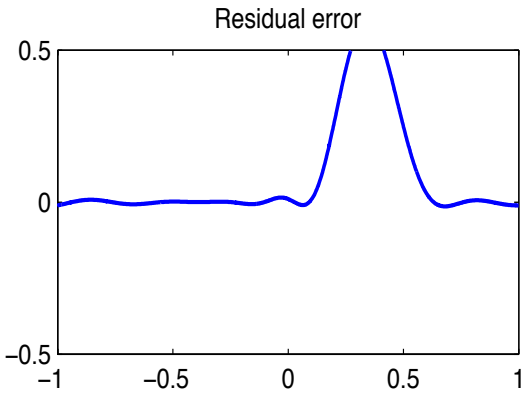
Runge's example $f(x) = \frac{1}{1+25x^2}$

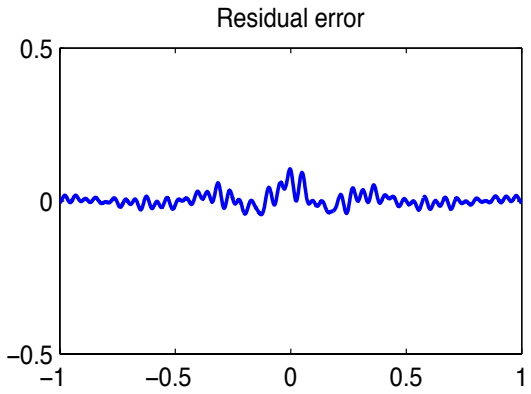
Weights: $w_j = 1 + |j|$
20 Interpolation points chosen uniformly at random from $[-1, 1]$.

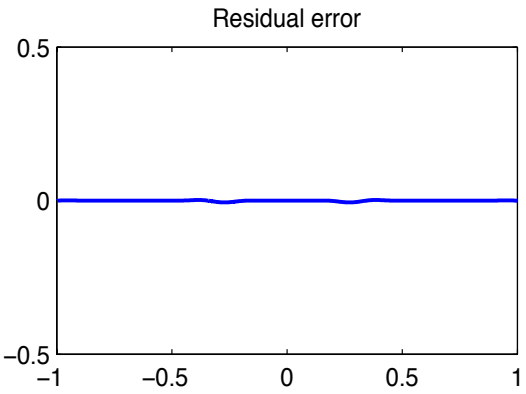




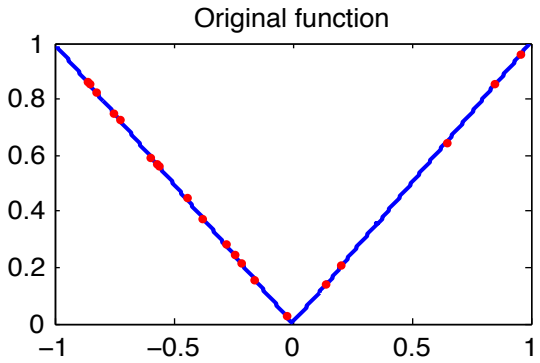








Numerical example II for the trigonometric system



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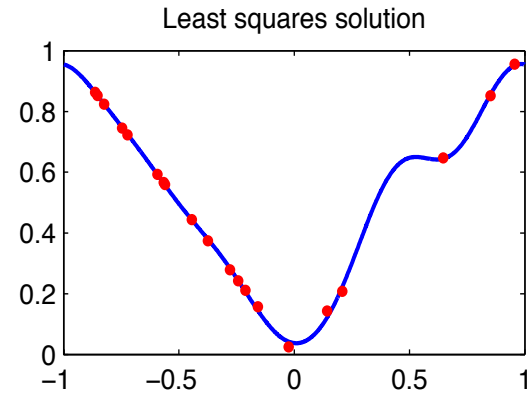

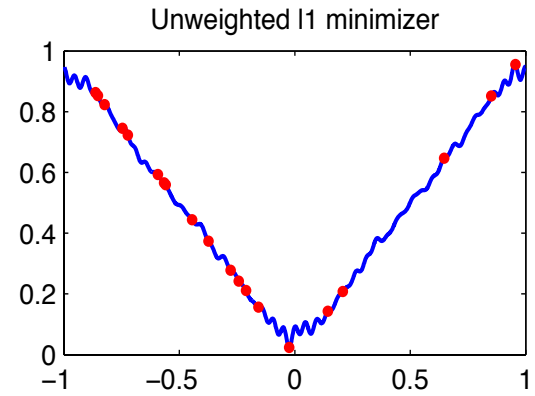

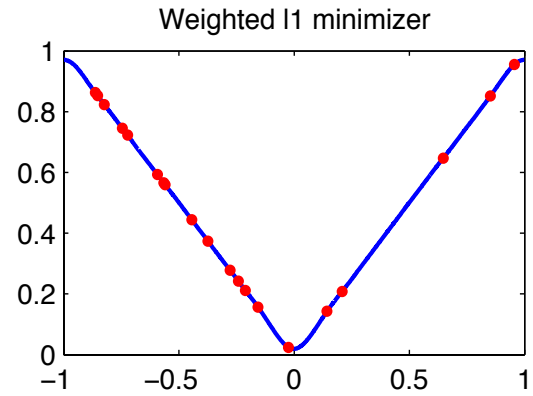

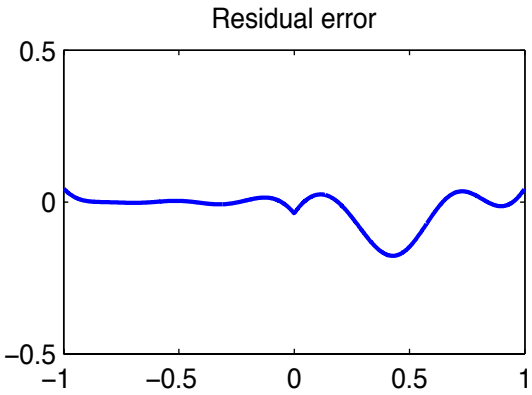

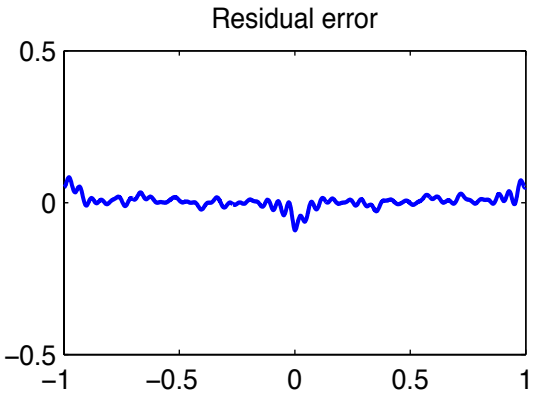

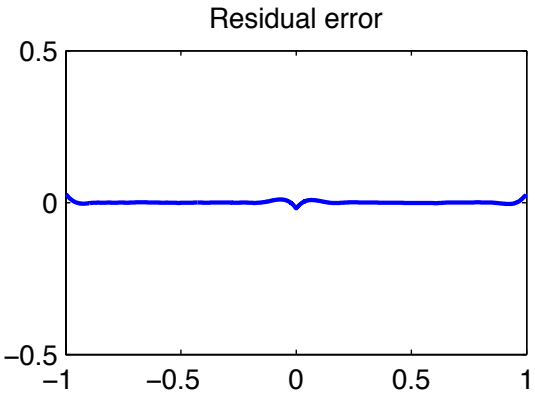

Chebyshev polynomials

Chebyshev-polynomials C_j, $j = 0, 1, 2, \ldots$

$$\int_{-1}^{1} C_j(x) C_k(x) \frac{dx}{\pi\sqrt{1-x^2}} = \delta_{j,k}, \quad j, k \in \mathbb{N}_0,$$

$\|C_0\|_\infty = 1$ and $\|C_j\|_\infty = \sqrt{2}$.

Stable recovery of polynomials that are s-sparse in the Chebyshev system via unweighted ℓ_1-minimization from

$$m \geq Cs \log^3(s) \log(N)$$

samples drawn i.i.d. from the Chebyshev measure $\frac{dx}{\pi\sqrt{1-x^2}}$.

Also error guarantees in $\|\cdot\|_{\omega,1}$ via $\ell_{\omega,1}$-minimization.

Legendre polynomials

Legendre polynomials L_j, $j = 0, 1, 2, \ldots$

$$\frac{1}{2}\int_{-1}^{1} L_j(x)L_k(x)dx = \delta_{j,k}, \quad \|L_j\|_\infty \leq \sqrt{j+1}, \quad j, k \in \mathbb{N}_0.$$

Unweighted Case: $K = \max_{j=0,\ldots,N-1} \|L_j\|_\infty = \sqrt{N}$ leads to bound

$$m \geq CK^2 s \log^3(s) \log(N) = CNs \log^3(s) \log(N).$$

Preconditioning

Preconditioned system $Q_j(x) = v(x)L_j(x)$ with $v(x) = (\pi/2)^{1/2}(1-x^2)^{1/4}$ satisfies

$$\int_{-1}^{1} Q_j(x)Q_k(x)\frac{dx}{\pi\sqrt{1-x^2}} = \delta_{j,k}, \quad \|Q_j\|_\infty \leq \sqrt{3}, \quad j,k \in \mathbb{N}_0.$$

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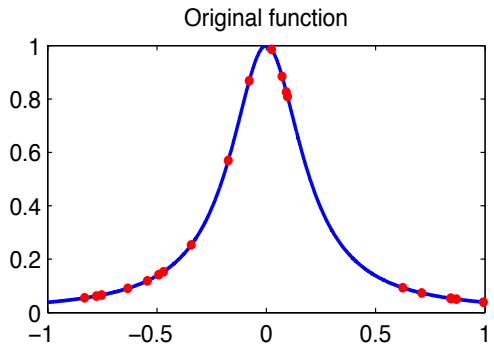
samples drawn i.i.d. from the Chebyshev measure $\frac{dx}{\pi\sqrt{1-x^2}}$.

Alternatively, use weight $\omega_j = \sqrt{j+1}$ and uniform or Chebyshev measure.

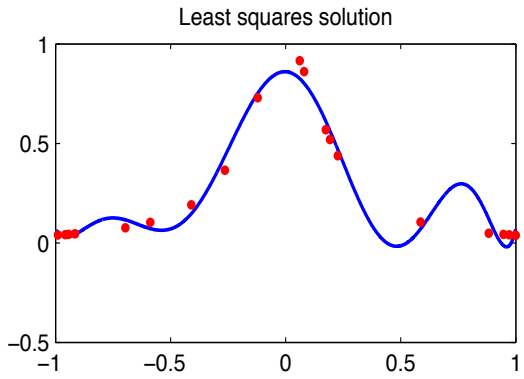
Numerical example for Chebyshev polynomials

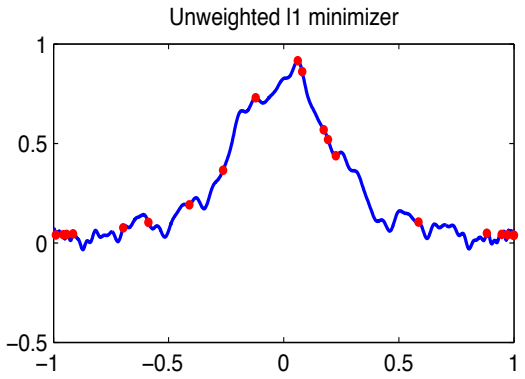


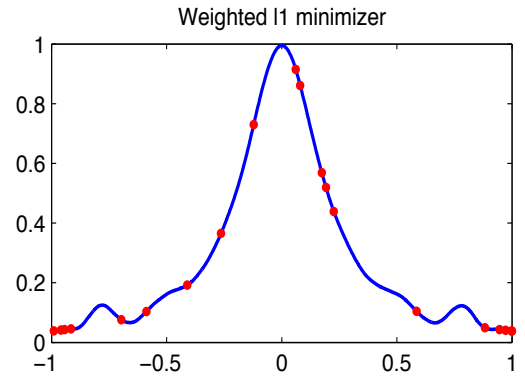
$f(x) = \frac{1}{1+25x^2}$

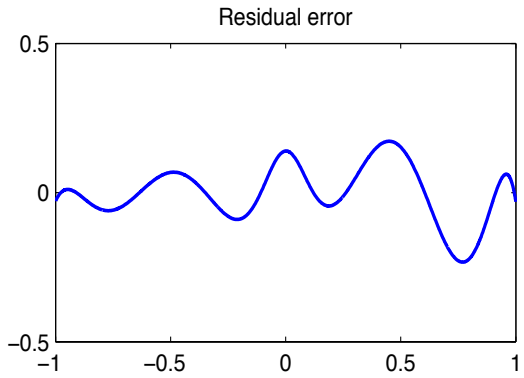
Weights: $w_j = 1 + j$.

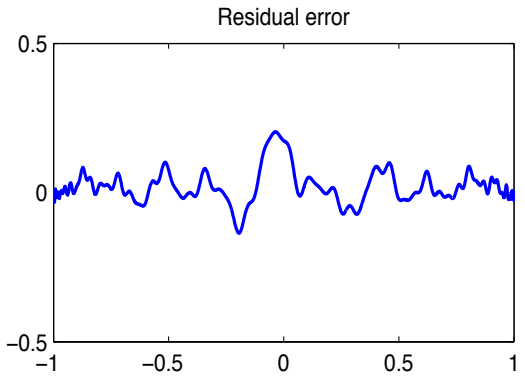
20 Interpolation points chosen i.i.d. at random according to Chebyshev measure $d\nu(x) = \frac{dx}{\pi\sqrt{1-x^2}}$.

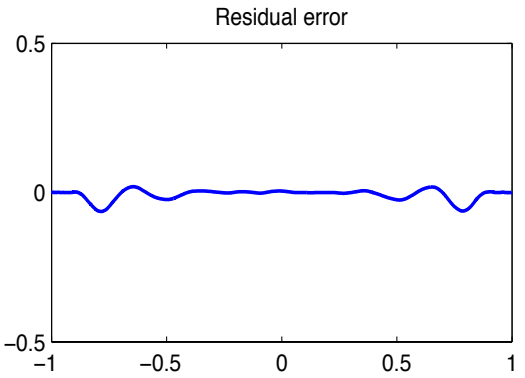




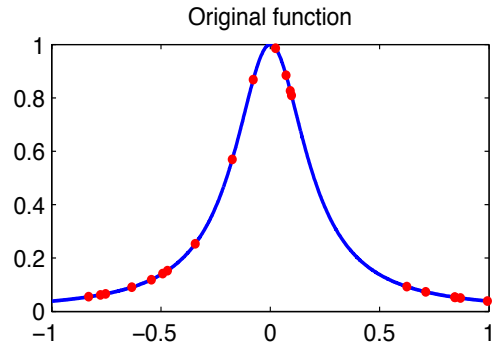








Numerical example for Legendre polynomials



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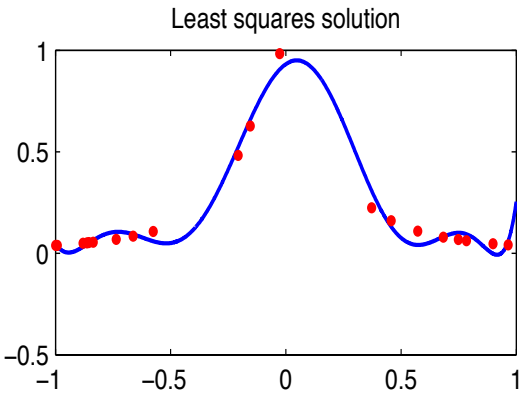

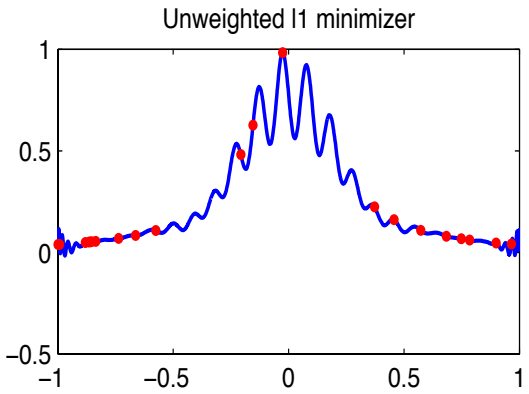

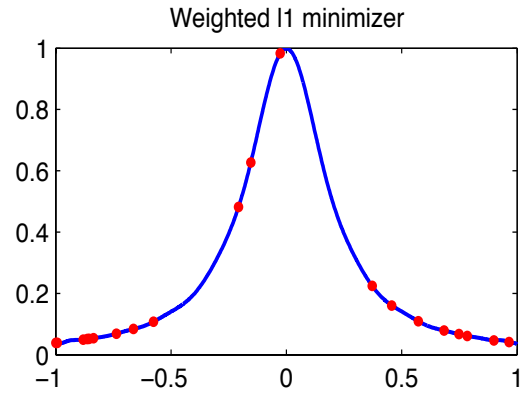

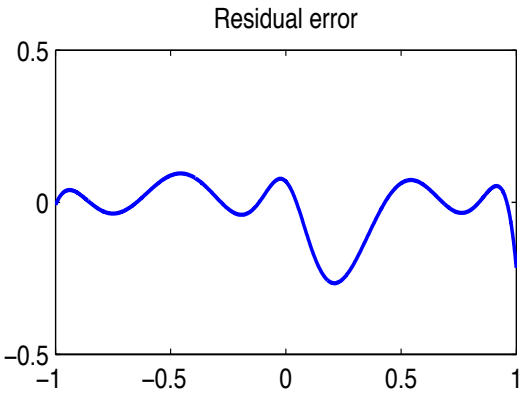

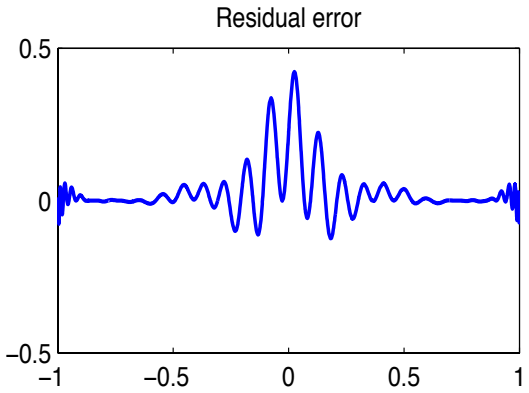

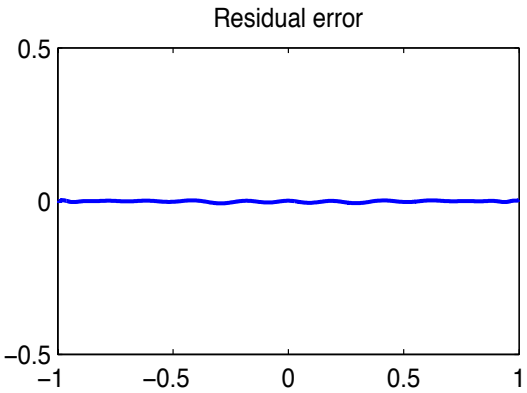

Spherical harmonics

Y_ℓ^k, $-k \le \ell \le k$, $k \in \mathbb{N}_0$: orthonormal system in $L^2(S^2)$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} Y_\ell^k(\phi,\theta)\overline{Y_{\ell'}^{k'}(\theta,\phi)}\sin(\theta)d\phi d\theta = \delta_{\ell,\ell'}\delta_{k,k'}$$

$(\phi,\theta) \in [0,2\pi) \times [0,\pi)$: spherical coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\phi) \end{pmatrix} \in S^2$$

With ultraspherical polynomials p_n^α:

$$Y_\ell^k(\phi,\theta) = e^{ik\phi}(\sin\theta)^{|k|}p_{\ell-|k|}^{|k|}(\cos\theta), \quad (\phi,\theta) \in [0,2\pi) \times [0,\pi)$$

Unweighted RIP for Spherical Harmonics

L^∞-bound: $\|Y_\ell^k\|_\infty \leq k^{1/2}$

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Preconditioning I (Krasikov '08)
With $w(\theta, \phi) = |\sin(\theta)|^{1/2}$

$$\|wY_\ell^k\|_\infty \leq Ck^{1/4}.$$

Unweighted RIP for Spherical Harmonics

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Preconditioning I (Krasikov '08)
With $w(\theta,\phi) = |\sin(\theta)|^{1/2}$

$$\|wY_\ell^k\|_\infty \leq Ck^{1/4}.$$

Preconditioning II (Burq, Dyatkov, Ward, Zworski '12)
With $v(\theta,\phi) = |\sin^2(\theta)\cos(\theta)|^{1/6}$,

$$\|vY_\ell^k\|_\infty \leq Ck^{1/6}.$$

unweighted RIP for associated preconditioned random sampling matrix $\frac{1}{\sqrt{m}}A \in \mathbb{C}^{m\times N}$ with sampling points drawn according to $\nu(d\theta, d\phi) = v^{-2}(\theta,\phi)\sin(\theta)d\theta d\phi = |\tan(\theta)|^{1/3}d\theta d\phi$ with high probability if

$$m \geq CsN^{1/6}\log^4(N).$$

Weighted RIP for spherical harmonics

Y_ℓ^k, $-k \leq \ell \leq k$, $k \in \mathbb{N}_0$: spherical harmonics.

Recall:
L^∞-bound: $\| Y_\ell^k \|_\infty \leq k^{1/2}$
Preconditioned L^∞-bound for $v(\theta, \phi) = |\sin^2(\theta)\cos(\theta)|^{1/6}$:

$$\| v Y_\ell^k \|_\infty \leq C k^{1/6}.$$

Weighted RIP for spherical harmonics

Y_ℓ^k, $-k \leq \ell \leq k$, $k \in \mathbb{N}_0$: spherical harmonics.

Recall:
L^∞-bound: $\|Y_\ell^k\|_\infty \leq k^{1/2}$
Preconditioned L^∞-bound for $v(\theta, \phi) = |\sin^2(\theta)\cos(\theta)|^{1/6}$:

$$\|vY_\ell^k\|_\infty \leq Ck^{1/6}.$$

Weighted RIP:
With weights $\omega_{k,\ell} \geq k^{1/6}$ the preconditioned random sampling matrix $\frac{1}{\sqrt{m}}A \in \mathbb{C}^{m\times N}$ satisfies $\delta_{\omega,s} \leq \delta$ with high probability if

$$m \geq C\delta^{-2}s\log(s)\log(N).$$

Comparison of error bounds

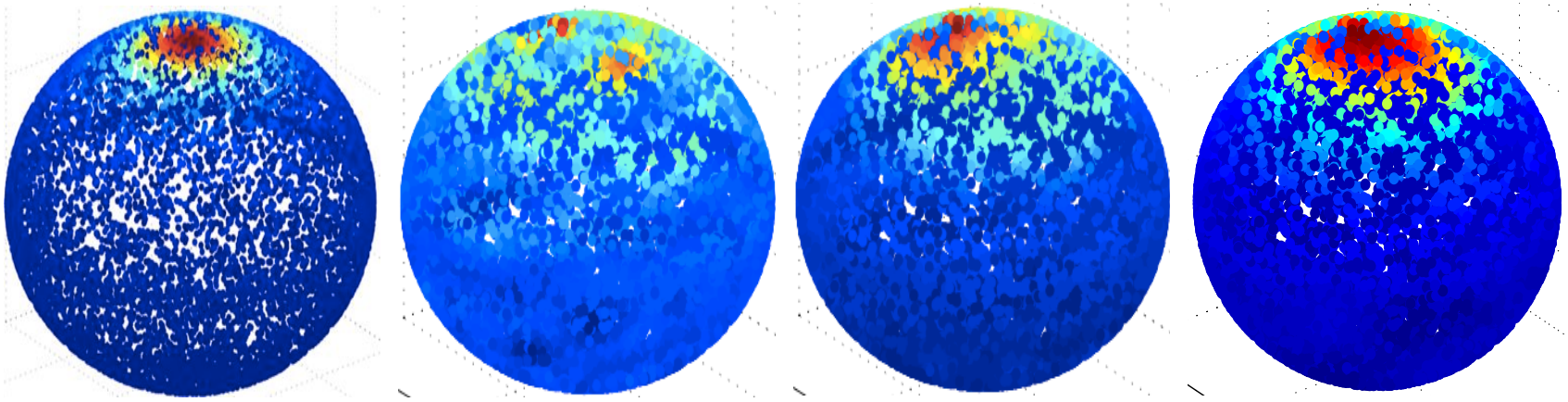
Error bound for reconstruction of $f \in A_{\omega,p}$ from $m \geq Cs\log^3(s)\log(N)$ samples drawn i.i.d. at random from the measure $\nu(d\theta, d\phi) = |\tan(\theta)|^{1/3}$ via weighted ℓ_1-minimization:

$$\|f - f^\sharp\|_\infty \leq |||f - f^\sharp|||_{\omega,1} \leq Cs^{1-1/p}|||f|||_{\omega,p}, \quad 0 < p < 1.$$

Compare to error estimate for unweighted ℓ_1-minimization:
If $m \geq CN^{1/6}s\log^3(s)\log(N)$ then

$$|||f - f^\sharp|||_1 \leq Cs^{1-1/p}|||f|||_p, \quad 0 < p < 1.$$

Numerical Experiments for Sparse Spherical Harmonic Recovery



Original function unweighted ℓ_1 $\omega_{k,\ell} = k^{1/6}$ $\omega_{k,\ell} = k^{1/2}$

Original function $f(\theta, \phi) = \frac{1}{|\theta|^2 + 1/10}$

High dimensional function interpolation

Tensorized Chebyshev polynomials on $\mathcal{D} = [-1,1]^d$

$$C_{\mathbf{k}}(\mathbf{t}) = C_{k_1}(t_1)C_{k_2}(t_2)\cdots C_{k_d}(t_d), \quad \mathbf{k} \in \mathbb{N}_0^d$$

with C_k the L^2-normalized Chebyshev polynomials on $[-1,1]$. Then

$$\frac{1}{2^d}\int_{[-1,1]^d} C_{\mathbf{k}}(\mathbf{t})C_{\mathbf{j}}(\mathbf{t})d\mathbf{t} = \delta_{\mathbf{k},\mathbf{j}}, \quad \mathbf{j},\mathbf{k} \in \mathbb{N}_0^d.$$

Expansions $f(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{N}_0^d} x_{\mathbf{k}} C_{\mathbf{k}}(\mathbf{t})$ with $\|x\|_p < \infty$ for $0 < p < 1$ and large d (even $d = \infty$) appear in parametric PDE's (Cohen, DeVore, Schwab 2011, ...).

(Weighted) sparse recovery for tensorized Chebyshev polynomials

L^∞-bound: $\|C_{\mathbf{k}}\|_\infty = 2^{\|\mathbf{k}\|_0/2}$.
Curse of dimension: Classical RIP bound requires

$$m \geq C 2^d s \log^3(s) \log(N).$$

(Weighted) sparse recovery for tensorized Chebyshev polynomials

L^∞-bound: $\|C_{\mathbf{k}}\|_\infty = 2^{\|\mathbf{k}\|_0/2}$.
Curse of dimension: Classical RIP bound requires

$$m \geq C 2^d s \log^3(s) \log(N).$$

Weights: $\omega_{\mathbf{j}} = 2^{\|\mathbf{k}\|_0/2}$.
Weighted RIP bound:

$$m \geq C s \log^3(s) \log(N)$$

Approximate recovery requires $x \in \ell_{\omega,p}$!

Comparison Classical Interpolation vs. Weighted ℓ_1-minimization

Classical bound

$$\|f - \widetilde{f}\|_\infty \leq C m^{-r/d} \|f\|_{C^r}$$

Interpolation via ℓ_1-minimization

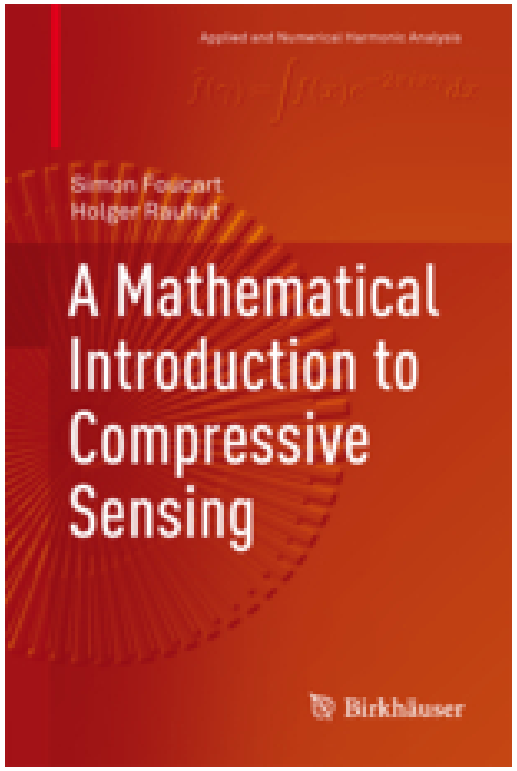
$$\|f - \widetilde{f}\|_\infty \leq C \left(\frac{m}{\ln^4(m)} \right)^{1-1/p} \|\!| f |\!\|_{\omega,p}, \quad 0 < p < 1.$$

Better rate if $1/p - 1 > r/d$, i.e.,

$$p < \frac{1}{r/d + 1}.$$

For instance, when $r = d$, then $p < 1/2$ is sufficient.

Avertisement



S. Foucart, H. Rauhut,
A Mathematical Introduction to Compressive Sensing
Applied and Numerical Harmonic Analysis, Birkhäuser, 2013.

Thank you!