Analysis of Compressive Sensing in Radar

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Overview

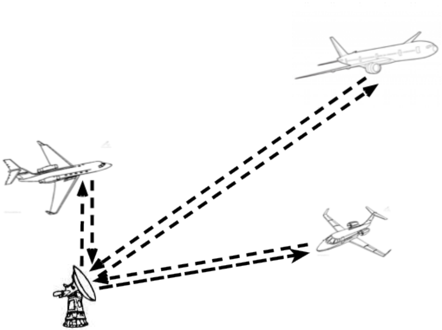
Several radar setups with compressive sensing approaches

- Range-Doppler resolution via compressive sensing
- Sparse MIMO Radar
- Antenna arrays with randomly positioned antennas

Time-Frequency Structured Random Matrices

Resolution of Range-Doppler in Radar

Resolution of Range-Doppler



Received signal is superposition of delayed and modulated (Doppler shifted) versions of sent signal.

Task: Determine delays (corresponding to distances; range) and Doppler shifts (corresponding to radial speed) from subsampled receive signal!

Gabor Systems in Finite Dimensions

Translation and Modulation on $\mathbb{C}^m$

$$(T^k g)_j = g_{(j-k) \mod m} \quad \text{and} \quad (M^\ell g)_j = e^{2\pi i \ell j/m} g_j.$$

Time-frequency shifts

$$\pi(\lambda) = M^\ell T^k, \quad \lambda = (k, \ell) \in \{0, \ldots, m-1\}^2.$$

For $g \in \mathbb{C}^m$ define Gabor synthesis matrix ($\omega = e^{2\pi i/m}$)

$$\Psi_g = (\pi(\lambda) g)_{\lambda \in \{0,\ldots,m-1\}^2}$$

$$= \left(\begin{array}{cccc|ccc|cc} g_0 & g_{m-1} & \cdots & g_1 & g_0 & \cdots & g_1 & \cdots & g_1 \\ g_1 & g_0 & \cdots & g_2 & \omega g_1 & \cdots & \omega g_2 & \cdots & \omega^{m-1} g_2 \\ g_2 & g_1 & \cdots & g_3 & \omega^2 g_2 & \cdots & \omega^2 g_3 & \cdots & \omega^{2(m-1)} g_3 \\ g_3 & g_2 & \cdots & g_4 & \omega^3 g_3 & \cdots & \omega^3 g_4 & \cdots & \omega^{3(m-1)} g_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ g_{m-1} & g_{m-2} & \cdots & g_0 & \omega^{m-1} g_{m-1} & \cdots & \omega^{m-1} g_0 & \cdots & \omega^{(m-1)^2} g_0 \end{array} \right).$$

Use of $\Psi_g \in \mathbb{C}^{m \times m^2}$ as measurement matrix in compressive sensing

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Radar model (Herman, Strohmer 2008)

Emitted signal: $g \in \mathbb{C}^m$.
Objects scatters g and radar device receives the contribution

$$x_\lambda \pi(\lambda) g = x_{k,\ell} M^\ell T^k g.$$

T^k corresponds to delay, i.e., distance of object
M^ℓ corresponds to Doppler shift, i.e., speed of the object
$x_{k,\ell}$ reflectivity of object

Received signal is superposition of contribution of all scatteres:

$$y = \sum_{\lambda \in \Lambda} x_\lambda \pi(\lambda) g = \Psi_g x.$$

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We will choose g as random vector below.

Reconstruction via compressive sensing

Reconstruction of x from $y = Ax$ via ℓ_1-minimization

$$\min \|z\|_1 \quad \text{subject to } Az = y$$
$$\min \|z\|_1 \quad \text{subject to } \|Az - y\|_2 \leq \eta$$

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Alternatives:
Matching Pursuits
Iterative hard thresholding (pursuit)
Iteratively reweighted least squares
...

Uniform vs. nonuniform recovery

Often recovery results are for random matrices $A \in \mathbb{R}^{m \times N}$; choose generator $g \in \mathbb{C}^m$ for Ψ_g at random

- Uniform recovery

  With high probability on A every sparse vector is recovered;

$$\mathbb{P}(\forall s\text{-sparse } x,\ \text{recovery of } x \text{ is successful using } A) \geq 1 - \varepsilon.$$

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  Recovery conditions on A

  - Tangent cone (descent cone) of norm at x intersects $\ker A$ trivially.
  - Dual certificates

Restricted isometry property (RIP)

Definition

The restricted isometry constant δ_s of a matrix $A \in \mathbb{C}^{m \times N}$ is defined as the smallest δ_s such that

$$(1 - \delta_s)\|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1 + \delta_s)\|\mathbf{x}\|_2^2$$

for all s-sparse $\mathbf{x} \in \mathbb{C}^N$.

Stable and robust recovery

Theorem (Candès, Romberg, Tao '04 – Cai, Zhang '13)

Let $A \in \mathbb{C}^{m \times N}$ *with* $\delta_{2s} < 1/\sqrt{2} \approx 0.7071$. *Let* $\mathbf{x} \in \mathbb{C}^N$, *and assume that noisy data are observed,* $\mathbf{y} = A\mathbf{x} + \mathbf{e}$ *with* $\|\mathbf{e}\|_2 \leq \tau$. *Let* $\mathbf{x}^\#$ *by a solution of*

$$\min_{\mathbf{z}} \|\mathbf{z}\|_1 \quad \text{such that} \quad \|A\mathbf{z} - \mathbf{y}\|_2 \leq \tau.$$

Then

$$\|\mathbf{x} - \mathbf{x}^\#\|_2 \leq C\frac{\sigma_s(\mathbf{x})_1}{\sqrt{s}} + D\tau,$$
$$\|\mathbf{x} - \mathbf{x}^\#\|_1 \leq C\sigma_s(\mathbf{x})_1 + D\sqrt{s}\tau$$

for constants $C, D > 0$, *that depend only on* δ_{2s}. *Here*

$$\sigma_s(\mathbf{x})_1 = \inf_{z:\|z\|_0 \leq s} \|\mathbf{x} - \mathbf{z}\|_1.$$

Implies exact recovery in the s-sparse and noiseless case.

Dual certificate

Theorem (Fuchs 2004, Tropp 2005)

For $A \in \mathbb{C}^{m \times N}$, $\mathbf{x} \in \mathbb{C}^N$ with support S is the unique solution of

$$\min \|\mathbf{z}\|_1 \quad \textit{subject to } A\mathbf{z} = A\mathbf{x}$$

if A_S is injective and there exists a dual vector $\mathbf{h} \in \mathbb{C}^m$ such that

$$(A^*\mathbf{h})_j = \operatorname{sgn}(x_j), \quad j \in S, \qquad\qquad |(A^*\mathbf{h})_\ell| < 1, \quad \ell \in \overline{S}.$$

Corollary

Let $\mathbf{a_1}, \ldots, \mathbf{a}_N$ be the columns of $A \in \mathbb{C}^{m \times N}$. For $\mathbf{x} \in \mathbb{C}^N$ with support S, if the matrix A_S is injective and if

$$|\langle A_S^\dagger \mathbf{a}_\ell, \operatorname{sgn}(\mathbf{x}_S) \rangle| < 1 \quad \textit{for all } \ell \in \overline{S},$$

then the vector $\mathbf{x}$ is the unique ℓ_1-minimizer with $\mathbf{y} = A\mathbf{x}$.

Here, $A_S^\dagger$ is Moore-Penrose pseudo inverse.

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One ingredient: Check that $\|A_S^* A_S - I\|_{2\to 2} \le \delta < 1$.

Stability and robustness via dual certificate

Theorem

Let $\mathbf{x} \in \mathbb{C}^N$ *and* $A \in \mathbb{C}^{m \times N}$ *with* ℓ_2*-normalized columns. Denote by* $S \subset [N]$ *the indices of the* s *largest absolute entries of* $\mathbf{x}$*. Assume that*

(i) *there is a* *dual certificate* $\mathbf{u} = A^* \mathbf{h} \in \mathbb{C}^N$ *with* $\mathbf{h} \in \mathbb{C}^m$ *s.t.*

$$\mathbf{u}_T = \operatorname{sgn}(\mathbf{x})_T, \quad \|\mathbf{u}_{T^c}\|_\infty \leq \frac{1}{2}, \quad \|\mathbf{h}\|_2 \leq 3\sqrt{s}.$$

(ii) $\|A_T^* A_T - I\|_{2 \to 2} \leq \frac{1}{2}$.

Given noisy measurements $\mathbf{y} = A\mathbf{x} + \mathbf{e} \in \mathbb{C}^m$ *with* $\|\mathbf{e}\|_2 \leq \tau$*, the solution* $\hat{\mathbf{x}} \in \mathbb{C}^N$ *of noise-constrained* ℓ_1*-minimization satisfies*

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_2 \leq 52\sqrt{s}\tau + 16\sigma_s(\mathbf{x})_1.$$

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Remark: Error bound is worse by factor of $\sqrt{s}$ than the one obtained from RIP.

Can be removed again by additionally requiring the weak RIP.

Random choice of generator g

Recall Gabor synthesis matrix

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Choice of g as subgaussian random vector:
Entries of g are independent, mean-zero, variance one and subgaussian: $\mathbb{P}(|g_j| \geq t) \leq 2e^{-Kt^2}$ for some $K > 0$.
Examples:

- Rademacher: entries ± 1 with equal probability
- Steinhaus: entries are uniformly distributed on complex torus $\{z \in \mathbb{C} : |z| = 1\}$
- Gaussian: entries are standard real or complex Gaussian variables

RIP estimate for random generator (Krahmer, Mendelson, Rauhut 2014)

Theorem

Let $\Psi_g \in \mathbb{C}^{m \times N}$, $N = m^2$, be generated by a subgaussian random vector g. If, for $\delta \in (0,1)$,

$$m \geq C\delta^{-2} s \max\{\log^2 s \log^2 N, \log(\varepsilon^{-1})\},$$

then with probability at least $1-\varepsilon$ the restricted isometry constant of $\frac{1}{\sqrt{m}}\Psi_g$ satisfies $\delta_s \leq \delta$.

Implies stable and robust recovery via ℓ_1 minimization with high probability if $m \geq Cs \log^2(s) \log^2(N)$.

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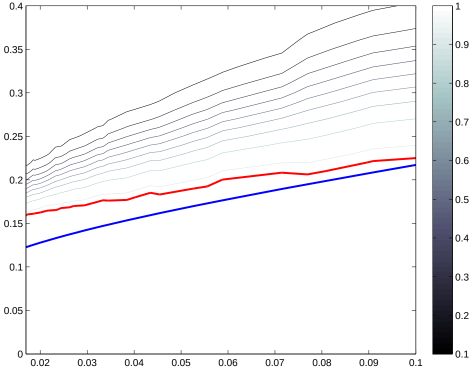
Previous results:

Pfander, R, Tropp 2012: $m \geq C_\delta s^{3/2} \log^3 N$

Nonuniform recovery, Pfander, R 2010: $m \geq Cs \log(N)$

Theorem can be generalized to certain other systems of operators (instead of time-frequency shifts).

Numerical experiments for Steinhaus g



Horizontal axis $1/m = m/m^2$, vertical axis s/m.
Contours of success probability, 93% success rate, $1/(2\log(m))$.
Numerical experiments suggest $s \le \frac{m}{2\log(m)}$ ensures s-sparse recovery.

Proof ingredient: chaos processes

Recall: δ_s is smallest constant such that

$$(1-\delta_s)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta_s)\|x\|_2^2$$

Equivalently, with $T_s = \{x \in \mathbb{C}^N : \|x\|_2 \le 1, \|x\|_0 \le s\}$

$$\delta_s = \sup_{x \in T_s} \left| \|Ax\|_2^2 - \|x\|_2^2 \right|$$

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In our case

$$Ax = \frac{1}{\sqrt{m}} \Psi_g x = \frac{1}{\sqrt{m}} \sum_{k,\ell=0}^{m-1} x_{k,\ell} M^\ell T^k g = V_x g,$$

with $V_x = \frac{1}{\sqrt{m}} \sum_{k,\ell=0}^{m-1} x_{k,\ell} M^\ell T^k$. Since entries of g have mean zero and variance one, $\mathbb{E}\|V_x g\|_2^2 = \|x\|_2^2$, so that

$$\delta_s = \sup_{x \in T_s} |\|V_x g\|_2^2 - \mathbb{E}\|V_x g\|_2^2|$$

This is a second order chaos processes.

Generic Chaining for Chaos Processes

Theorem (Krahmer, Mendelson, R 2014)

Let $\mathcal{B} = -\mathcal{B} \subset \mathbb{C}^{m \times N}$ be a symmetric set of matrices and $\xi \in \mathbb{C}^N$ be a subgaussian random vector. Then

$$\mathbb{E} \sup_{B \in \mathcal{B}} \left| \|B\xi\|_2^2 - \mathbb{E}\|B\xi\|_2^2 \right| \\ \leq C_1 \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2})^2 + C_2 \Delta_{\|\cdot\|_F}(\mathcal{B}) \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}).$$

Here, $\|B\|_F = \sqrt{\mathrm{tr}(B^*B)}$ denotes the Frobenius norm.

Symmetry assumption $\mathcal{B} = -\mathcal{B}$ can be dropped at the cost of slightly more complicated bound.

Here, $\Delta_{\|\cdot\|}(\mathcal{B})$ is the diameter of $\mathcal{B}$ with respect to $\|\cdot\|$ and $\gamma_2(\mathcal{B}, \|\cdot\|)$ is Talagrand's γ_2-functional which can be bounded by

$$\gamma_2(\mathcal{B}, \|\cdot\|) \leq C \int_0^{\Delta_{\|\cdot\|}(\mathcal{B})} \sqrt{\log N(\mathcal{B}, \|\cdot\|, u)}\, du,$$

where $N(\mathcal{B}, \|\cdot\|, u)$ are the covering numbers of $\mathcal{B}$ at radius u.

Tail bound

Theorem (Krahmer, Mendelson, R '14 – Dirksen '15)

Let $\mathcal{B} = -\mathcal{B} \subset \mathbb{C}^{m \times N}$ and $\xi \in \mathbb{C}^N$ be a subgaussian random vector. Then

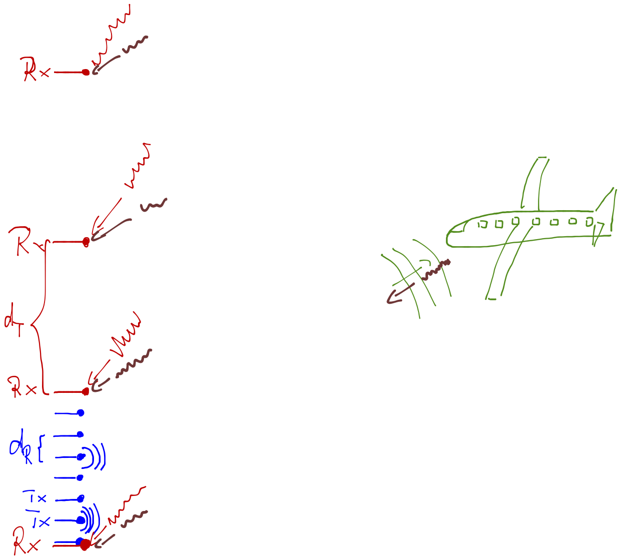
$$\mathbb{P}\left(\sup_{B \in \mathcal{B}} \left| \|B\xi\|_2^2 - \mathbb{E}\|B\xi\|_2^2 \right| \geq C_1 E + t \right)$$
$$\leq 2 \exp\left(-C_2 \min\left\{ \frac{t^2}{V^2}, \frac{t}{U} \right\} \right),$$

where

$$E := \Delta_{\|\cdot\|_F}(\mathcal{B}) \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2}) + \gamma_2(\mathcal{B}, \|\cdot\|_{2\to 2})^2,$$
$$V := \Delta_{\|\cdot\|_{2\to 2}} \Delta_{\|\cdot\|_F}(\mathcal{B}),$$
$$U := \Delta^2_{\|\cdot\|_{2\to 2}}(\mathcal{B}).$$

Sparse MIMO radar

MIMO Radar



MIMO Radar in 2D

- N_T transmit antennas at locations

$$\big(0, (k-1)d_T\lambda\big), \quad k = 1, 2, \dots, N_T$$

- N_R receive antennas at locations

$$\big(0, (j-1)d_R\lambda\big), \quad j = 1, \dots, N_R$$

Choose $d_T = 1/2$, $d_R = N_T/2$.

Then system has similar characteristics as antenna array with $N_T N_R$ antennas.

(Alternatively, $d_T = N_R/2$, $d_R = 1/2$)

Measurement model
Strohmer, Friedlander 2012; Yu, Petropulu, Poor, 2011

- Transmit antennas send periodic continuous-time complex Gaussian pulses $s_1, \ldots, s_{N_T}$ with period T and band-width B.

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- Transmit antennas send periodic continuous-time complex Gaussian pulses $s_1, \dots, s_{N_T}$ with period T and band-width B.
- Echo of target of unit reflectivity at position $(r\cos(\theta), r\sin(\theta)$ and radial speed v at receiver j:

$$r_j(t) = \sum_{k=1}^{N_T} e^{2\pi i c\lambda^{-1}(t - d_{k,j}(t)/c)} s_k(t - d_{k,j}(t)/c)$$

with carrier frequency λ, speed of light c, and distance from kth transmitter to target and from target to jth receiver

$$d_{k,j}(t) = 2(r + vt) + \sin(\theta) d_T (k-1)\lambda + \sin(\theta)(j-1) d_R \lambda$$

- Demodulation (multiplication of $r_j(t)$ with $e^{-2\pi i c\lambda^{-1} t}$) and assuming $B \ll \lambda$ (narrowband transmit waveforms), $v \ll c$ (slowly moving targets), $r \gg \lambda N_R N_T/2$ (far field scenario) yields measurements

$$y_j(t) \approx e^{2\pi i \cdot 2\lambda^{-1} r} e^{2\pi i \sin(\theta) d_R (j-1)} \sum_{k=1}^{N} e^{2\pi i \cdot 2\lambda^{-1} vt} e^{2\pi i \sin(\theta) d_T (k-1)} s_k(t - 2r/c)$$

Discretization

- By the Shannon-Nyquist sampling theorem, the band-limited periodic complex Gaussian transmit signals can be represented by their sampled counterparts $\mathbf{s}_k \in \mathbb{C}^{N_t}$ (sampled over one period $[0, T]$); N_t: number of samples.

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- Target is described by triple (θ, r, v) (azimuth, range, velocity); Discretization of $(\beta, \tau, f) = (\sin(\theta), 2r/c, 2\lambda^{-1}v)$ with stepsizes

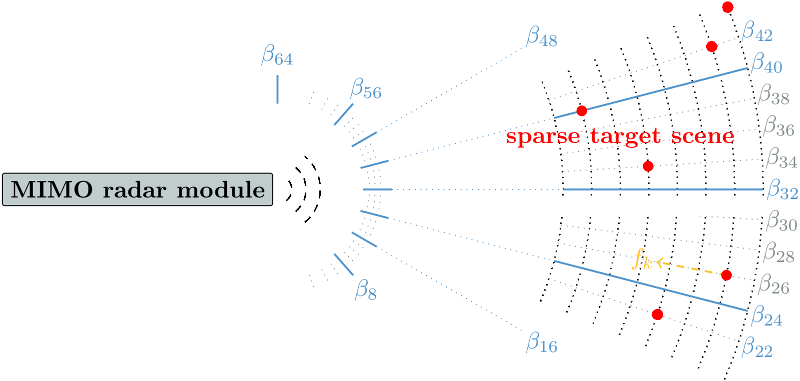
$$\Delta_\beta = \frac{2}{N_T N_R}, \quad \Delta_\tau = \frac{1}{2B}, \quad \Delta_f = \frac{1}{T}$$

yields grid

$$\mathcal{G} = \left\{(\beta\Delta_\beta, \tau\Delta_\tau, f\Delta_f) : \beta \in [N_R N_T], \tau \in [N_t], f \in [N_t]\right\}$$

Index set $G = [N_T N_R] \times [N_t] \times [N_t]$ of size $N := N_R N_T N_t^2$ (here $[k] = \{1, \ldots, k\}$).

Discretization grid



$N_R = N_T = 8$

Measurement Model I

One target with unit reflectivity at grid point indexed by $\Theta = (\beta, \tau, f) \in G$

Discrete time samples at receiver j ($j = 1, \dots, N_R$)

$$\mathbf{y}_j = (y_j(\Delta_t), y_j(2\Delta_t), \dots, y_j(N_t\Delta_t))^T$$

$$= e^{2\pi i \cdot c\lambda^{-1}\tau\Delta_\tau} \left[e^{2\pi i \cdot d_R\beta\Delta_\beta(j-1)} \sum_{k=1}^{N_T} e^{2\pi i \cdot d_T\beta\Delta_\beta(k-1)} \mathbf{M}_f \mathbf{T}_\tau \mathbf{s}_k \right] \in \mathbb{C}^{N_t}$$

with translation and modulation operators on $\mathbb{C}^{N_t}$ defined as

$$(\mathbf{T}_\tau \mathbf{s})_k = \mathbf{s}_{k-\tau}, \qquad (\mathbf{M}_f \mathbf{s})_k = e^{2\pi i \cdot \frac{fk}{N_t}} (\mathbf{s})_k.$$

Targets on grid points index by $\Theta \in G$ with reflectivities ρ_Θ, setting also $x_\Theta = e^{2\pi i \cdot c\lambda^{-1}\tau\Delta_\tau} \rho_\Theta$; measurements at receiver j:

$$\mathbf{y}_j = \sum_{\Theta \in G} x_\Theta \underbrace{e^{2\pi i \cdot d_R\beta\Delta_\beta(j-1)} \sum_{k=1}^{N_T} e^{2\pi i \cdot d_T\beta\Delta_\beta(k-1)} \mathbf{M}_f \mathbf{T}_\tau \mathbf{s}_k}_{=:\mathbf{A}_\Theta^j}$$

Measurement Model II

Collection of sampled signals at all receivers:

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{N_R} \end{pmatrix} = \begin{pmatrix} \sum_{\Theta \in G} x_\Theta \mathbf{A}_\Theta^1 \\ \vdots \\ \sum_{\Theta \in G} x_\Theta \mathbf{A}_\Theta^{N_R} \end{pmatrix} = \mathbf{A}\mathbf{x} \in \mathbb{C}^{N_r \cdot N_t}$$

Measurement matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_\Theta^1 \\ \vdots \\ \mathbf{A}_\Theta^{N_R} \end{pmatrix}_{\Theta \in G} \in \mathbb{C}^{N_R N_t \times N_R N_T N_t^2}, \quad G = [N_R N_T] \times [N_t] \times [N_t]$$

$$\mathbf{A}_\Theta^j = e^{2\pi i \cdot d_R \beta \Delta_\beta (j-1)} \sum_{k=1}^{N_T} e^{2\pi i \cdot d_T \beta \Delta_\beta (k-1)} \mathbf{M}_f \mathbf{T}_\tau \mathbf{s}_k \in \mathbb{C}^{N_t}, \ \Theta = (\beta, \tau, f)$$

Structured random matrix; the $\mathbf{s}_1, \dots, \mathbf{s}_{N_T}$ are independent subgaussian random vectors, e.g. standard complex Gaussian random vectors, Rademacher vectors, or Steinhaus vectors

Number of measurements: $m = N_R N_t$, signal dimension $N = N_R N_T N_t^2$, i.e., $m \ll N$; recall $d_T = 1/2$, $d_R = N_T/2$, $\Delta_\beta = \frac{2}{N_T N_R}$

Reconstruction via Compressive Sensing

Reconstruction problem of solving $\mathbf{Ax} = \mathbf{y}$ is underdetermined.

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Use compressive sensing for reconstruction!

For recovery, we will study ℓ_1-minimization

$$\min_{\mathbf{z}} \|\mathbf{z}\|_1 \quad \text{subject to } \|\mathbf{Ax} - \mathbf{y}\|_2 \leq \tau$$

and LASSO

$$\min_{\mathbf{z}} \frac{1}{2}\|\mathbf{Az} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{z}\|_1$$

Recovery for random support sets

Strohmer and Friedlander (2013) showed recovery of the correct support via (debiased) LASSO for s-sparse signals with random support (and random signs) with high probability under the condition

$$m = N_R N_t \geq Cs \log(N)$$

(plus minor additional technical assumptions).

Proof is based on an analysis of the coherence of $\mathbf{A}$ and a general recovery result for random signals due to Tropp (2008).

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Question:
Can we avoid the assumption of randomness of the support?

The RIP for MIMO radar measurements

Theorem (Dorsch, R 2015)

If

$$N_t \geq C\delta^{-2} s \max\{\log^2(s)\log^2(N), \log(\varepsilon^{-1})\}$$

then the rescaled random radar measurement matrix $\frac{1}{\sqrt{N_R N_T N_t}}\mathbf{A} \in \mathbb{C}^{N_R N_t \times N_R N_T N_t^2}$ *satisfies* $\delta_s \leq \delta$ *with probability at least* $1 - \varepsilon$.

Implies stable and robust sparse recovery via ℓ_1-minimization.

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Compared to other random matrix constructions in compressed sensing (where $m \asymp s\log(eN/s)$) the result requires more measurements because here $m = N_t N_R$; i.e., we suffer an additional factor of N_R.

Almost optimality of RIP estimate

Theorem (Dorsch, R 2015)

If a realization of the random MIMO radar measurement matrix $\frac{1}{\sqrt{N_R N_T N_t^2}}\mathbf{A}$ *satisfies* $\delta_s \leq 0.7$ *for* $s \leq N_t^2$*, then necessarily*

$$N_t \geq Cs\log(eN_t^2/s).$$

Proof idea:
Introduce $S_\beta := \{(\beta', \tau', f') \in G : \beta' = \beta\}$.
If $\mathbf{x}$ has support in S_β then one can write

$$\mathbf{Ax} = \mathbf{a}_R(\beta) \otimes \mathbf{Bx}_{S_\beta}$$

for a vector $\mathbf{a}_R(\beta) \in \mathbb{C}^{N_R}$ with entries of magnitude 1 and a matrix $\mathbf{B} \in \mathbb{C}^{N_t \times N_t^2}$. Applying lower sparse recovery bounds for $\mathbf{B}$ yields the claim.

Towards nonuniform recovery

Recovery depends on the fine structure of the support set:

Equivalence class of angles, $\beta, \beta' \in [N_R N_T]$,

$$\beta \sim \beta' : \beta' - \beta \equiv 0 \mod N_R$$

This definition is motivated by the fact that the columns of $\mathbf{A}$ satisfy, for $\Theta = (\beta, \tau, f), \Theta' = (\beta', \tau', f')$,

$$\langle \mathbf{A}_\Theta, \mathbf{A}_{\Theta'} \rangle = N_R \widehat{\delta}_{\beta,\beta'} = \begin{cases} N_R & \text{if } \beta \sim \beta', \\ 0 & \text{otherwise.} \end{cases}$$

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Intuitively, the more elements of the support S are such that the corresponding β's are contained in different equivalent classes the better the matrix $\mathbf{A}_S$ is conditioned.

Well-balanced support sets

For a support set $S \subset G = [N_R N_T] \times [N_t] \times [N_t]$ let

$$S_{[\beta]} := \{\Theta' = (\beta', \tau', f') \in S : \beta' \sim \beta\}.$$

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Definition

A support set $S \subset G$ is called η-balanced, if for all angle classes $[\beta]$,

$$|S_{[\beta]}| \leq \eta \frac{|S|}{N_R}.$$

The parameter η ranges in $[1, N_R]$.
A small value of η means that the support S is well-distributed over the angle classes, which is favorable for recovery.

Nonuniform recovery I

Theorem

Let $\mathbf{x} \in \mathbb{C}^N$ *and* $S \subset G$ *be an index set corresponding to* s *largest absolute entries in* $\mathbf{x}$*. Assume* S *to be* η*-balanced and that the signs of the coefficients* $\mathbf{x}_S$ *form a Steinhaus sequence. Assume measurements* $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n} \in \mathbb{C}^{N_R N_t}$ *are given, where the signals* $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_T}$ *generating the measurement matrix* $\mathbf{A}$ *are independent subgaussian random vectors, and* $\|\mathbf{n}\|_2 \leq \tau$*. If*

$$m = N_R N_t \gtrsim \eta s \log^3(N/\varepsilon),$$

then, with probability at least $1 - \varepsilon$*, the solution* $\mathbf{x}^\#$ *to constrained* ℓ_1*-minimization satisfies*

$$\|\mathbf{x}^\# - \mathbf{x}\|_2 \leq C_1 \sigma_s(\mathbf{x})_1 + C_2 \frac{\tau\sqrt{s}}{\sqrt{N_T N_R N_t}},$$

where C_1 *and* C_2 *are numerical constants.*

Exact recovery for s-sparse scene $\mathbf{x}$.

Nonuniform recovery for LASSO

Theorem (Dorsch, R 2015)

Let $\mathbf{x} \in \mathbb{C}^N$*,* $N = N_R N_T N_t^2$*, be a fixed* s*-sparse target scene with* η*-balanced support* S *such that the phases of the nonzero entries form a random Steinhaus sequence and such that*

$$\min_{\Theta \in S} > 8\sigma \sqrt{\frac{2\log(N)}{N_T N_R N_t}}.$$

Draw $\mathbf{A}$ *at random and let* $\mathbf{y} = \mathbf{A} + \mathbf{e}$ *be noisy measurements with random noise,* $\mathbf{e} \sim \mathcal{CN}(0, \sigma^2)$*. Assume that*

$$m = N_R N_t \geq C\eta s \log^3(N/\varepsilon).$$

Then, with probability at least $1 - 7\max\{\varepsilon, N^{-3}\}$*, the solution* $\mathbf{x}^\sharp$ *of*

$$\min_{\mathbf{z}} \frac{1}{2}\|\mathbf{Az} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{z}\|_1 \qquad \text{with } \lambda = 2\sigma\sqrt{\frac{2\log(N)}{N_T N_R N_t}}$$

satisfies $\mathrm{supp}(\mathbf{x}) = \mathrm{supp}(\mathbf{x}^\#)$*.*

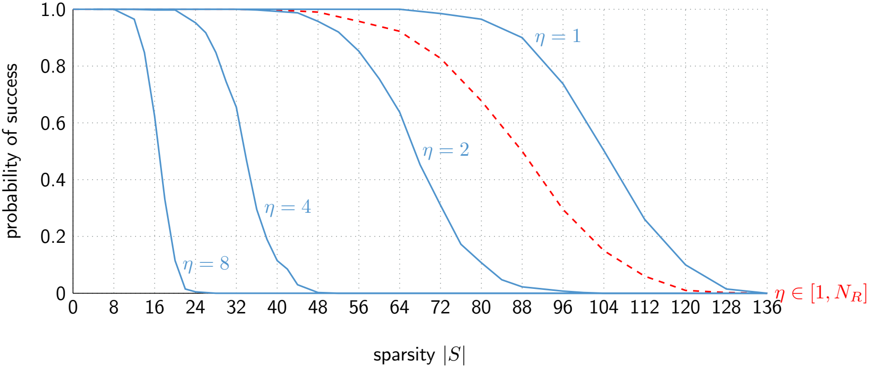
Remarks about nonuniform recovery

- The debiased LASSO estimator $\widehat{\mathbf{x}}$ – least squares on $\mathrm{supp}(\mathbf{x}^\sharp)$, after computing LASSO solution – satisfies

$$\|\mathbf{x} - \widehat{\mathbf{x}}\|_2 \leq 2\sigma\sqrt{2s\log(N)/(N_T N_R N_t)}.$$

- The randomness in the signs of the nonzero entries of $\mathbf{x}$ can likely be removed.
- For optimal balancedness parameter $\eta = 1$, we obtain a (near-)optimal bound on the number of measurements: $m \geq Cs\log^3(N/\varepsilon)$.
- RIP-result covers the worst case where $\eta = N_R$.
- A random support set will be η-balanced for small η with high probability, which explains the result of Strohmer and Friedlander.

Numerical experiments for Doppler-free scenario



Success rates for various values of η

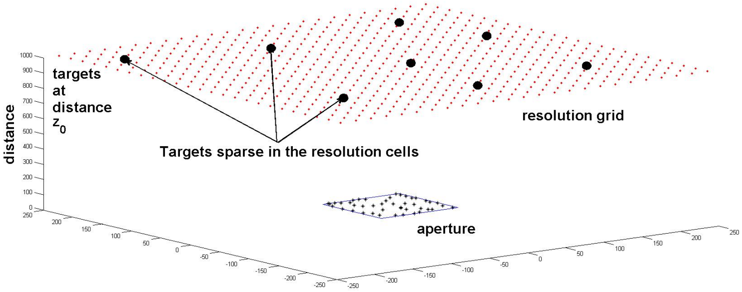
red curve corresponds to randomly chosen support sets

$N_T = N_R = 8$ transmit and receive antennas, $N_t = 64$ time-domain samples, grid size $N = N_T N_R N_t = 4096$

$m = N_R N_t = 512$ measurements

Antenna arrays with random antenna positions

Radar setup



n antenna elements on square $[0, B]^2$ in plane $z = 0$.

Targets in the plane $z = z_0$ on grid of resolution cells $r_j \in [-L, L]^2 \times \{z_0\}$, $j = 1, \dots, N$ with mesh size h.

$\mathbf{x} \in \mathbb{C}^N$: vector of reflectivities in resolution cells $(r_j)_{j=1,\dots,N}$.

Sensing mechanism (Fannjiang, Strohmer, Yan 2010)

Antenna at position $a \in \mathbb{R}^3$ emits monochromatic wave (wavelength λ, wavenumber ω) with amplitude at position $r \in \mathbb{R}^3$ given by Green's function of Helmholtz equation

$$H(a, r) = \frac{\exp\left(2\pi i \|r - a\|_2/\lambda\right)}{4\pi\|r - a\|_2}.$$

Approximation (valid for large z_0): $H(a, r) \approx \frac{e^{i\omega z_0}}{4\pi z_0} G(a, r)$ with

$$G(a, r) = \exp\left(\frac{i\omega}{2z_0}(|r_1 - a_1|^2 + |r_2 - a_2|^2)\right)$$

Signal corresponding to emitting antenna a_ℓ and receive antenna a_k (Born approximation)

$$y_{(k,\ell)} = \sum_{j=1}^{N} x_j G(a_\ell, r_j) G(r_j, a_k) = (A\mathbf{x})_{(k,\ell)}, \quad k, \ell = 1, \ldots, n$$

n^2 measurements

Random scattering matrix

Choose antenna positions a_j, $j \in [n]$, independently and uniformly at random in $[0, B]^2$. Then $A \in \mathbb{C}^{n^2 \times N}$ is structured random matrix. Entries

$$A_{(k,\ell);j} = G(a_k, r_j)G(r_j, a_\ell), \quad (k,\ell) \in [n]^2, j \in [N].$$

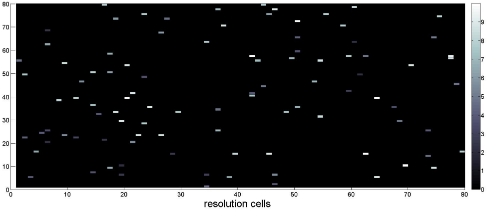
Define $v(a_k, a_\ell) = (G(a_k, r_j)G(r_j, a_\ell))_{j\in[N]} \in \mathbb{C}^N$. Then

$$A = \begin{pmatrix} v(a_1, a_1) \\ v(a_1, a_2) \\ \vdots \\ v(a_2, a_1) \\ \vdots \\ v(a_n, a_n) \end{pmatrix}$$

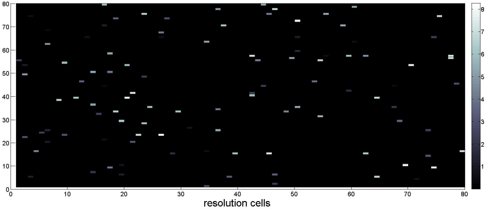
Rows and columns are coupled.
Under the condition $\frac{hB}{\lambda z_0} \in \mathbb{N}$ we have $\mathbb{E}A^*A = I$.

Reconstruction via ℓ_1-minimization

Sparse scene (sparsity $s = 100$, 6400 grid points):



Reconstruction ($n = 30$ antennas, 900 noisy measurements, SNR 20dB)



Nonuniform recovery

Theorem (Hügel, R, Strohmer 2014)

Let $\mathbf{x} \in \mathbb{C}^N$. Choose the n antenna positions independent and uniformly at random in $[0, B]^2$. Assume $\frac{hB}{\lambda z_0} \in \mathbb{N}$, where h is mesh size and λ the wavelength; further

$$n^2 \geq Cs \ln^2(N/\varepsilon) \, .$$

Let $\mathbf{y} = A\mathbf{x} + \mathbf{e} \in \mathbb{C}^{n^2}$ with $\|\mathbf{e}\|_2 \leq \eta n$. Let $\mathbf{x}^\#$ be the solution to

$$\min \|\mathbf{z}\|_1 \quad \textit{subject to} \quad \|\mathbf{y} - A\mathbf{z}\|_2 \leq \eta n.$$

Then with probability at least $1 - \varepsilon$

$$\|\mathbf{x} - \mathbf{x}^\#\|_2 \leq C_1 \sigma_s(\mathbf{x})_1 + C_2 \sqrt{s} \eta.$$

Exact recovery when $\eta = 0$ and $\sigma_s(\mathbf{x})_1 = 0$.
RIP estimate open.

Conclusions

Analysis of compressive sensing in various radar setups may be interesting and challenging!

- Time-Frequency (range-Doppler) structured random matrices (Pfander, R 2010; Pfander, R, Tropp 2012; Krahmer, Mendelson, R - 2014)
- MIMO radar with random transmit pulses (Friedlander, Strohmer 2014; Dorsch, R 2015)
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  - ...
- More challenging mathematical problems from radar applications
  - Off grid compressive sensing
  - ...

The End

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