Hints for solving the exercises in Chapter 1

Hints to Exercise 1.3 Each polynomial of degree $n \geq 1$ has exactly n roots (counted according to their multiplicity, respectively).

Hints to Exercise 1.4 For $j = 0, 1, \ldots, n+1$ apply the Lagrange interpolation formula to the function $f(x) = x^j$. For $1 \leq j \leq n$, the result has to be evaluated at the point $x = 0$. In the situation $j = n+1$, use an error representation for polyomial interpolation.

Hints to Exercise 1.5 Represent the interpolating polynomial w.r.t. to the Lagrange basis as well as w.r.t. the Newton basis, and compare the leading coefficients.

Hints to Exercise 1.6 For both interpolating polynomials consider the expansion w.r.t. the Newton basis and compare the leading coefficients.

Hints to Exercise 1.7 According to the Weierstraß approximation theorem, for each number $\delta > 0$ the function f can be approximated by a polynomial q with $(*)\ \| f - q \|_\infty \leq \delta$. Consider then $P = q + \sum_{k=0}^{n} c_k L_k$ where q is a polynomial satisfying $(*)$ for a sufficiently small number $\delta > 0$, and the coefficients $c_0, \ldots, c_n$ are appropriately chosen. Here, $L_0, \ldots, L_n$ as usual denote the Lagrange polynomials.

Hints to Exercise 1.8 part (a), $\Longleftarrow$: consider the linear mapping $A : \mathcal{V} \to \mathbb{R}^{n+1},\ v \mapsto (\varphi_j(v))_{j=0}^{n}$.

Hints to Exercise 1.9 (a) Consider for $f \in C^{n+1}[a,b]$ the Taylor expansion $f(x) = p_n(x) + r_{n+1}(x)$ with

$$p_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j, \qquad r_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

For $\mathcal{L}_n r_{n+1}$ consider the Lagrange interpolation formula.

(b) Apply the Lagrange interpolation formula. For "$\geq$" fix a point x and consider a function $f \in C[a,b]$ with $\| f \|_\infty = 1$ and $f(x_j) = \mathbf{sgn} \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}$.

Hints to Exercise 1.10 (a) Use mathematical induction and a trigonometric theorem.

(b) Apply the rule of de L‘ Hospital.

(c) First consider $x \in (-1, 1)$ and compute the term $\frac{d}{d\theta}\cos(n\theta)$ by using the representation $\cos\theta = x \in (-1, 1)$. The statement for the boundary points $x = \pm 1$ then follows immediately.