# Hints for solving the exercises in Chapter 10

**Hints to Exercise 10.1** This exercise is an extention of Lemma 10.6. Proceed as in the proof of Lemma 8.15 on the boundedness of the sequence of matrices $A, A^2, \ldots$ . For "(i) $\Longrightarrow$ (ii)", restrict the considerations to matrices $\mathcal{H} \in \mathbb{R}^{N\times N}$ with $r_\sigma(\mathcal{H}) = 1$.

The Exercises 10.2−10.4 can be used to derive convergence results for the relaxation method, for example.

**Hints to Exercise 10.2** Use that sums and products of nonnegative matrices are again nonnegative.

**Hints to Exercise 10.3** For the solution of this exercise one needs a correspondence between the eigenvalues of the matrix $A^{-1}P$ and those of $B^{-1}P$. Additionally apply the theorem of Perron.

**Hints to Exercise 10.4** Follows immediately from Exercise 10.3 and the estimate $r_\sigma(A) \le r_\sigma(B)$ for matrices $A,\ B \in \mathbb{R}^{N\times N}$ with $0 \le A \le B$ (see Theorem 9.15).

**Hints to Exercise 10.5** For the matrix $A$ and for *all* admissable sets of indices $\mathcal{K},\ \mathcal{L}$ one has to determine indices $k \in \mathcal{K},\ \ell \in \mathcal{L}$ with $a_{k\ell} \ne 0$. For the matrix $B$, admissable sets of indices $\mathcal{K},\ \mathcal{L}$ have to be determined such that $b_{k\ell} = 0$ holds for all $k \in \mathcal{K},\ \ell \in \mathcal{L}$.

**Hints to Exercise 10.6** For the verification of the implication "$\Longrightarrow$" one has (depending on the situation $k < j$ or $k > j$ or $k = j$) to find a connecting chain. For the verification of the implication "$\Longleftarrow$", consider a chain $k_0, k_1, \ldots, k_M \in \{1, \ldots, N\}$ connecting the indices $k$ and $k+1$. By assumption $a_{k,k_\ell} \ne 0 \Longrightarrow k_\ell \in \{k-1, k, k+1\}$. From this the required inequality $a_{k,k+1} \ne 0$ can be verified. Analogously one proceeds for indices $k$ and $k-1$.

**Hints to Exercise 10.7** From the properties $a_{(r-1)(M-1),r(M-1)} \ne 0$ for $r = 2, 3, \ldots, M-1$ and $a_{r(M-1),r(M-1)} \ne 0$ for $r = 1, 2, \ldots, M-1$, a chain connecting $k$ and $j$ can be determined. Additionally use the fact that each diagonal subblock has tridiagonal form.

**Hints to Exercise 10.8** (a) For an eigenvector $x \in \mathbb{C}^N$ corresponding to the eigenvalue $\lambda$ of the matrix $A$ consider an index $k$ with $|x_k| = \|x\|_\infty$ and estimate (by using the diagonal dominance of $A$) the number $|\lambda - a_{kk}|$, such that $\mathbf{Re}\ \lambda \ge 0$ is obtained and that $\mathbf{Re}\ \lambda = 0$ implies $\lambda = 0$. Finally show that this case $\lambda = 0$ cannot occur.

(b) Use linear algebra.

**Hints to Exercise 10.9** This is an immediate consequence of Theorem 10.33 on the characterisation for M−matrices and the corresponding nonnegative Neumann expansion.

**Hints to Exercise 10.10** "(i) $\Longrightarrow$ (ii)" follows immediately from Theorem 9.15, and for the verification of "(i) $\Longrightarrow$ (iii)" one has to determine a real number $s \ge 0$ such that $B = sI - A \ge 0$ is satisfied. The implication "(iii) $\Longrightarrow$ (i)" follows also with Theorem 9.15, and for "(iii) $\Longrightarrow$ (iv)" a correspondence between the eigenvalues of $A$ and those of $B$ is needed. Finally, for the verification of "(iv) $\Longrightarrow$ (iii)" a number $s \ge 0$ has to be determined such that $B = sI - A \ge 0$ is satisfied.

**Hints to Exercise 10.11** The linear system of equations to be considered here is the same as in Exercise 10.13.

(a) One has to check if the assumptions of Theorem 10.33 are satisfied. For the verification of $r_\sigma(D^{-1}(L+R)) < 1$ it is sufficient to show that the matrix is $A$ irreducibly diagonally dominant (see the proof of Theorem 10.19).

(b) Show first that the function $\theta$ solves the given boundary value problem for the right−hand side $\varphi \equiv 1$. The required inequality then follows with well-known error estimates for central differences of first and second order.

(c) The procedure here is the same as in the proof of Theorem 9.9. Addionally an upper bound of the form $M \ge \|\theta\|_\infty$ has to be determined.

(d) A multiplication of $I = A^{-1}A$ with the error $v_* - z$ gives the solution.

**Hints to Exercise 10.12** Here a regular decomposition $A = B_\omega + P_\omega$ has to be determined which depends on $\omega$, and $B_\omega$ and $P_\omega$ are a lower and upper triangular matrix, resp. An application of Exercise 10.4 with an appropriate choice of $\omega = \omega_k$ (with $B_k = B_{\omega_k}$ and $P_k = P_{\omega_k}$ for $k = 1, 2$) finally gives the solution.

**Hints to Exercise 10.13** (a) Apply the equivalence “(i) $\Longleftrightarrow$ (iv)” in Exercise 10.10, with $A$ replaced by $A + \gamma I$ and with an appropriate constant $\gamma$. A subsequent application of Exercise 10.9 gives the solution to this exercise.

(b) For this special situation the eigenvalues can be determined explicitly (Lemma 9.11), and an application of part (a) and an estimate of the form $\cos(x) \le p_2(x)$ for $|x| \le \pi/2$ with an appropriate polynomial $p_2 \in \Pi_2$ (can be obtained by a Taylor expansion at 0) gives the solution.

**Hints to Exercise 10.14** The eigenvalues of the matrix $A$ can be determined explicitly (Lemma 9.11). The symmetry can be used to find out if the matrix is positive definite or not. Similarly Lemma 9.11 can be applied to determine the eigenvalues of the matrix $\mathcal{H}_J = I - D^{-1}A$ and thus also the spectral radius. The optimal relaxation parameter $\omega_L$ coincides with the parameter $\omega_*$ considered in Theorem 10.43.

**Hints to Exercise 10.16** One has to find out what the form of the matrices $\widehat{D}$, $\widehat{L}$, $\widehat{R}$ in the composition $A = \widehat{D} + \widehat{L} + \widehat{R}$ is. This can be used to represent $\widehat{\mathcal{J}}(\alpha) = \alpha \widehat{D}^{-1}\widehat{L} + \alpha^{-1}\widehat{D}^{-1}\widehat{R}$. Then determine appropriate transformation matrices $\widehat{S}_\alpha \in \mathbb{R}^{N\times N}$ with $\widehat{\mathcal{J}}(\alpha) = \widehat{S}_\alpha \widehat{\mathcal{J}}(1) \widehat{S}_\alpha^{-1}$.

**Hints to Exercise 10.17** The procedure is similar to Section 10.7, including the choice of the transformation matrices $S_\alpha$ for solving part (c) of this exercise.