Hints for solving the exercises in Chapter 12

Hints to Exercise 12.1 First consider submatrices of the form

$$A_\ell \;=\; \begin{bmatrix} \delta_1 & \gamma_2 & & 0 \\ \beta_2 & \delta_2 & \ddots & \\ & \ddots & \ddots & \gamma_\ell \\ 0 & & \beta_\ell & \delta_\ell \end{bmatrix} \qquad \text{for} \quad \ell = 1, 2, \ldots, N \qquad (*)$$

and find out in which form the determinant $\det(A_{\ell+1})$ depends on $\det(A_\ell)$ and $\det(A_{\ell-1})$ (for $\ell = 1, 2, \ldots, N-1$).

(a) Use mathematical induction w.r.t. $\ell = 1, 2, \ldots, N$ to determine a correspondence between $\det(A_\ell - \lambda I_\ell)$ and $\det(B_\ell - \lambda I_\ell)$, where A_ℓ and B_ℓ are submatrices of A and B according to $(*)$. The solution to this exercise then follows for the special case $\ell = N$.

(b) Consider the matrix PAP with the special permutation matrix

$$P \;=\; \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{R}^{N\times N}.$$

(c) The answer to the first part of this problem can be found with part (a) of this exercise, and for the second part of the problem use mathematical induction w.r.t. $\ell = 1, 2, \ldots, N$ to find a representation of $\det(A_\ell)$.

Hints to Exercise 12.2 (a) The problem corresponding to the eigenvalues and vectors can be treated after a consideration of the matrix $A(I - 2vv^\top)$.

(b) First derive a representation for the entries of the matrix $vv^\top Dvv^\top$.

Hints to Exercise 12.3 For $\mu \in \sigma(A)$ the situation is clear, and for $\mu \notin \sigma(A)$ consider $(*)$ $(A - \mu I)^{-1}(D - \mu I)x$ and find out some results on the spectral norms of the matrices in $(*)$. Here, D denotes the diagonal matrix $\operatorname{diag}(d_1, d_2, \ldots, d_N)$. The rest follows by using the symmetry of the considered matrices.

Hints to Exercise 12.4 (a) Apply the theorem of Gershgorin to the matrix $C(\theta) := D_\theta^{-1}(A + \theta B)D_\theta$, where the notation $D_\theta = \operatorname{diag}(1, \theta^{1/N}, \theta^{2/N}, \ldots, \theta^{(N-1)/N})$ is used.

(b) Consider $B = (b_{kj}) \in \mathbb{R}^{N\times N}$ with $b_{N1} = 1$ and $b_{kj} = 0$ otherwise.

Hints to Exercise 12.5 The assumption means

$$|\lambda - a_{kk}| \quad \geq \quad \sum_{\substack{j=1 \\ j\neq k}}^{N} |a_{kj}| \qquad \text{for} \quad k \;=\; 1, 2, \ldots, N.$$

For an arbitrary eigenvector $0 \neq x \in \mathbb{C}^N$ with $Ax = \lambda x$ consider the index set $\mathcal{K} = \{ 1 \leq k \leq N \;:\; |x_k| = \|x\|_\infty \}$ and verify $\lambda \in \partial\mathcal{G}_k$ for all $k \in \mathcal{K}$ by using the diagonaldominance. Then show the following by making a contradictory assumption: if the matrix A is irreducible then the identity $\mathcal{K} = \{1, 2, \ldots, N\}$ holds.

Hints to Exercise 12.6 (a) From the symmetry of the matrix A it follows that all eigenvalues of A are real, i.e., $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$, and the corresponding eigenvectors $x_1, \ldots, x_N \in \mathbb{R}^N$ may assumed as mutually orthonormal, $x_k^\top x_j = \delta_{kj}$. An expansion $x = \sum_{j=1}^N a_j x_j$ and some elementary estimates then give the solution to the problem.

(b) One basically has to proceed as in part (a). Additionally one has to show that with the notation $J' = \{1, 2, \ldots, N\} \backslash J$ the following holds: $\sum_{j\in J'} |a_j|^2 = \inf_{z\in E(J)} \|x - z\|_2^2$.

Hints to Exercise 12.7 For the verification of the second identity in the exercise show first that in the first identity of the theorem of Courant/Fischer, the condition “$\dim \mathcal{L} \le j$” can be replaced by “$\dim \mathcal{L} = j$”. For this purpose, at one step of the corresponding proof for a subspace $\mathcal{L} \subset \mathbb{R}^N$ with $\dim \mathcal{L} \le j$ another subspace $M \subset \mathbb{R}^N$ has to be considered which satisfies $\mathcal{L} \subset M$ and $\dim M = j$, and the corresponding maximal Rayleigh quotients have to be compared. Then the system of sets $\{ \mathcal{L}^\perp \,:\, \mathcal{L} \subset \mathbb{R}^N$ is a linear subspace, $\dim \mathcal{L} = j \}$ has to be considered.

The first identity of the exercise can be obtained from the second identity of this exercise, applied to the matrix $(A + \gamma I)^{-1}$ for a sufficiently large number $\gamma > 0$.

Hints to Exercise 12.9 Consider the two identities of Exercise 12.7 with the special subspace $M = \text{span}\{ \mathbf{e}_k \,:\, k \le \lfloor N/2 \rfloor \}$.