Hints for solving the exercises in Chapter 14

Hints to Exercise 14.2 Follows with Theorem 14.5 applied to a sequence of functions which can be obtained by applying the following theorem:

Theorem *For each point $x \in [a,b]$ there exists a system of infinitely often differentiable functions $g_\varepsilon : [a,b] \to \mathbb{R}$ für $\varepsilon > 0$ with the following properties: $g_\varepsilon(t) = 0$ for $|t-x| \geq \varepsilon$, and $\int_a^b g_\varepsilon(t)\psi(t)\,dt \to \psi(x)$ $(\varepsilon \to 0)$ for each continuous function $\psi : [a,b] \to \mathbb{R}$.*

Verify the continuity of the Peano kernel. A proof of the theorem is not required.

Hints to Exercise 14.3 (a) Show first that the polynomial P can be represented as follows,

$$P(x) = \sum_{j=0}^{2} \big(f_j L_{j0}(x) + f_j' L_{j1}(x) \big)$$

where the following notations are used:

$$L_{j1} = \ell_{j1}, \qquad \ell_{j1}(x) = (x - x_j) \prod_{\substack{k=0 \\ k \neq j}}^{2} \Big(\frac{x - x_k}{x_j - x_k} \Big)^2 \qquad \text{for } j = 0, 1, 2$$

$$L_{j0} = \ell_{j0} - \ell_{j0}' L_{j1}, \qquad \ell_{j0}(x) = \prod_{\substack{k=0 \\ k \neq j}}^{2} \Big(\frac{x - x_k}{x_j - x_k} \Big)^2 \qquad \text{——— « ———}$$

For the verification of the representation for the corresponding quadrature formula stated in the exercise, one has to evaluate the integrals $\int_{-1}^{1} L_{j0}(x)\,dx$ und $\int_{-1}^{1} L_{j1}(x)\,dx$ for $j = 0, 1, 2$. Here, the computations can be simplified by verifying the identities $L_{11}(-x) = L_{11}(x)$ and $L_{21}(x) = -L_{01}(-x)$ as well as $L_{20}(x) = L_{00}(-x)$, where the second of the three identities can be obtained by a comparison of the Hermitian interpolation properties.

(d) Use the representation (14.3).