Hints for solving the exercises in Chapter 4

Hints to Exercise 4.3 (a) For "$\Longleftarrow$" proceed as for the square root method to compute a Cholesky factorization of symmetric, positiv definite matrices. For "$\Longrightarrow$" show that a LR factorization of A can be used to determine a LR factorization for each of the considered principal submatrices.

Hints to Exercise 4.4 (a) Use the positive definitness of the matrix A with a special well−known vector.

(b) For fixed indices i and j, apply the positive definitness of the matrix A with the vector

$$x \;=\; (x_k) \in \mathbb{R}^N \quad with\ x_k \;=\; \begin{cases} 1, & k \;=\; i, \\ 0, & k \neq i,\ k \neq j, \\ \alpha, & k \;=\; j, \end{cases}$$

with $\alpha \in \mathbb{R}$ arbitrary. From that conclude that no real solution α to the corresponding quadratic equation exists. This finally yields the solution to the problem.

(c) A contradictory assumption leads to a contradiction to the statement of part (b) of the exercise.

Hints to Exercise 4.5 Consider the decomposition $(*)\ A = LL^\top$ with the lower triangular matrix $L = (\ell_{ij}) \in \mathbb{R}^{N\times N}$. For fixed index $i \in \{ m+2, m+3, \ldots, N \}$, prove by mathematical induction w.r.t. $j \in \{ 1, 2, \ldots, i-m-1 \}$ that $\ell_{ij} = 0$ holds. (Use $(*)$ to derive necessary conditions for the numbers ℓ_{ij}).

Hints to Exercise 4.6 Consider the notations

$$U \;=\; \left[\; u_1 \;\middle|\; \ldots \;\middle|\; u_N \;\right], \qquad V \;=\; \left[\; v_1 \;\middle|\; \ldots \;\middle|\; v_N \;\right], \qquad \langle u, v \rangle_2 \;=\; u^\top v$$

and show first that the representation

$$Ax \;\overset{(*)}{=}\; \sum_{k=1}^{N} \sigma_k \langle x, u_k \rangle_2 v_k \qquad \text{for} \quad x \;\in\; \mathbb{R}^N$$

holds. This representation $(*)$ has to be applied several times in the sequel.

(a) Derive a formula for $\| A \|_2$ in terms of σ_1, and proceed similarly for $\| A^{-1} \|_2$ and σ_N.

(b) Consider $x = \sum_{k=1}^N \alpha_k u_k$ and find out under which conditions on the coefficients $\alpha_1,\ \alpha_2, \ldots, \alpha_N$ the identity $\| b \|_2 \;=\; \| A \|_2 \| x \|_2$ is satisfied. Similarly consider $\Delta x \;=\; \sum_{k=1}^N \beta_k u_k$ and find out under which conditions on the coefficients $\beta_1,\ \beta_2, \ldots, \beta_N$ the identity $\| \Delta x \|_2 = \| A^{-1} \|_2 \| \Delta b \|_2$ is satisfied. As a consequence from these properties, the solution to the third subproblem in part (b) is obtained.

(c) First reduce the problem and determine vectors $b \in \mathbb{R}^N$ so that $\| x \|_2 = \| A^{-1} \|_2 \| b \|_2$ holds.

Hints to Exercise 4.9 Consider the notations

$$A \;=\; \left[\; a_1 \;\middle|\; \ldots \;\middle|\; a_N \;\right], \qquad Q \;=\; \left[\; q_1 \;\middle|\; \ldots \;\middle|\; q_N \;\right], \qquad R \;=\; (r_{ij}).$$

and compute $|\det A|$ by using the identity $A = QR$. On the other hand there holds $a_j \;=\; \sum_{i=1}^j r_{ij} q_i$ with mutually orthonormal vectors $q_1,\ q_2, \ldots, q_i$. This can be used to determine lower bounds for the number $\| a_j \|_2$.

Hints to Exercise 4.10 In part (a) compute

$$(A + uv^\top) \Big(A^{-1} - \frac{A^{-1} u v^\top A{-}1}{1 + v^\top A^{-1} u} \Big).$$