Hints for solving the exercises in Chapter 6

Hints to Exercise 6.1 It is sufficient to consider monomials. This leads to a linear system of equations with corresponding matrix $A = (x_j^k)_{k,j=0,\dots,N-1} \in \mathbb{R}^{N\times N}$. Apply a result from Chapter 1 on polynomial interpolation to show that the transposed matrix $A^\top$ and therefore also the matrix A itself is regular.

Hints to Exercise 6.2 Consider a number $\overline{x}$ with $\overline{x} \neq x_j$ for $j = 0, 1, \dots, n$ and a polynomial $p \in \Pi_{2n+2}$ which satisfies

$$p(\overline{x}) = 1, \qquad p(x_j) = 0 \quad \text{for} \quad j = 0, 1, \dots, n, \qquad p(x) \geq 0 \quad \text{for} \quad x \in \mathbb{R}.$$

Hints to Exercise 6.3 Consider a Taylor expansion of the function f of order n (for n not being fixed yet) at the point $x_m = (a+b)/2$ and apply it for a representation of the numbers $\int_a^b f(x)\,dx$ and $f(a)$ and $f(b)$. Then consider the difference $\int_a^b f(x)\,dx - Qf$ and determine the coefficients a_0, a_1 and a_2, such that all term except for the remainders vanish. It turns out that this is only possible for $n \leq 3$.

Hints to Exercise 6.4 The interpolating trigonometric polynomial Lf should be of the form

$$(Lf)(x) \;=\; \sum_{k=-N}^{N} d_k\, e^{\mathrm{i}kx}.$$

Consider the function $p(x) := (Lf)(x)e^{\mathrm{i}Nx}$ to show that there is exactly one interpolating function Lf of the given form. Moreover show that $\int_0^{2\pi}(Lf)(x) = 2\pi d_0$ holds. Then determine the coefficient d_0 by applying the discrete inverse Fourier transform.

Hints to Exercise 6.5 For $r = 1$, the Euler–Maclaurin summation formula is of the form

$$\frac{g(0)}{2} + g(1) + \dots + g(N-1) + \frac{g(N)}{2} \;=\; \int_0^N g(t)\,dt + \frac{B_2(0)}{2}\big(g'(N) - g'(0)\big) + \frac{B_4(0)}{24} g^{(4)}(\xi)$$

with some intermediate number $0 < \xi < N$, with $B_2(0) = 1/6$ und $B_4(0) = -1/30$. This fomula applied with the function $g(x) = x^3$ yields the solution.

Hints to Exercise 6.6 Apply the substitution rule with $x = \cos\theta$. Additionally use that the trigonometric polynomials $\cos(kx),\ k = 0, 1, \dots$ are mutually orthogonal with respect to the standard inner product $\langle u, v\rangle_2 = \int_0^{2\pi} u(x)v(x)\,dx$.

Hints to Exercise 6.7 Proposal: for each of the four integrals, compute with a precision of 10^{-8} the numbers

$$\begin{array}{llll}
T_0 & & & \\
T_1 & T_{01} & & \\
T_2 & T_{12} & T_{012} & \\
T_3 & T_{23} & T_{123} & T_{0123} \\
T_4 & T_{34} & T_{234} & T_{1234} \\
\vdots & \vdots & \vdots & \vdots \\
T_{j_*-1} & T_{j_*-2,j_*-1} & T_{j_*-3,j_*-2,j_*-1} & T_{j_*-4,j_*-3,j_*-2,j_*-1} \\
T_{j_*} & T_{j_*-1,j_*} & T_{j_*-2,j_*-1,j_*} & T_{j_*-3,j_*-2,j_*-1,j_*} \\
 & & & \big|T_{j_*-3,\dots,j_*} - T_{j_*-4,\dots,j_*-1}\big|
\end{array}$$

Here $4 \leq j_* \leq 12$ denotes the smallest index with $\big|T_{j_*-3,\dots,j_*} - T_{j_*-4,\dots,j_*-1}\big| \leq \varepsilon := 10^{-8}$. In the case $\big|T_{j-3,\dots,j} - T_{j-4,\dots,j-1}\big| > \varepsilon$ for $j = 4, 5, \dots, 12$ let $j_* = 12$.