Hints for solving the exercises in Chapter 8

Hints to Exercise 8.1 "$\Longrightarrow$": Let $\nu \in \{0, 1, \ldots, p\}$ be fixed. The consistency order p in fact means $L[t^\nu, h] = \mathcal{O}(h^{p+1})$ for $h \to 0$. Note moreover that $L[t^\nu, h]$ is a polynomial in h of degree $\le \nu \le p$. These observations can be applied to derive the identity $L[t^\nu, h] = 0$.

"$\Longleftarrow$": This follows as an application of Lemma 8.16, so it remains to show that the conditions of this lemma are satisfied. This can be done by mathematical induction for $\nu = 0, 1, \ldots, p$ by using a Taylor expansion of the function $L[t^\nu, \cdot]$ at $h = 0$ (see also the proof of Lemma 8.16).

Hints to Exercise 8.2 The consistency orders follow immediately from Lemma 8.16. Nullstability of the parameter dependent multistep method follows after a determination of the roots of the corresponding characteristic polynomial of degree three (estimate one root; the other roots can be determined by dividision).

Hints to Exercise 8.3 Lemma 8.16 shows for a method of consistency order $2m+1$ that the linear system of equations $(*)$ $A(\alpha_0, \ldots, \alpha_m, -\beta_0, \ldots, -\beta_m)^\top = 0$ necessarily must be satisfied, with the matrix

$$A = \left[\begin{array}{c|c} (j^\nu)_{\substack{\nu=0,\ldots,2m+1\\ j=0,\ldots,m}} & (\nu j^{\nu-1})_{\substack{\nu=0,\ldots,2m+1\\ j=0,\ldots,m}} \end{array}\right] \in \mathbb{R}^{(2m+2)\times(2m+2)}.$$

This is a regular matrix in fact, which follows from the simple Hermite interpolation problem where function and its derivative are given at the support abscissas $0, 1, \ldots, m$ (see also the proof of Theorem 8.54). that there exists no method From this it follows which has consistency order $2m+1$. The number of methods with consistency order $2m$ can be determined easily from the fact that the number of considered equations in $(*)$ here is reduced by one.

Hints to Exercise 8.4 (a) Apply Theorem 8.56 to obtain the general real−valued solution.

(b) The solution of the first difference equation can be obtained by using Theorem 8.56. The corresponding coefficients have to be fitted to the given initial conditions The second and the third difference equation are inhomogen and of first order, respectively, and the corresponding solutions therefore can easily be obtained by mathematical induction. Theorem 8.56 again can be applied here. This finally yields the representation $u_m = p_m(t)$ with some prominent polynomials p_m.

Hints to Exercise 8.6 For fixed $\xi \in \mathbb{C}$, $Q(\xi, \cdot)$ is a polynomial of degree $\le m$. A factorization of Q then yields the representation

$$Q(\xi, h\lambda) = (\alpha_m + h\lambda\beta_m) \prod_{k=1}^{m} (\xi - \xi_j(h\lambda)),$$

where it is used that the leading coefficient of $Q(\xi, \cdot)$ is $\alpha_m + h\lambda\beta_m$. From the consistency and null stability we may assume without loss of generality that

$$\begin{aligned} \xi_1(h\lambda) &\to \xi_1(0) = 1, && \text{for } h\lambda \to 0, \\ \xi_k(h\lambda) &\to \xi_k(0) \ne 1 && \text{for } h\lambda \to 0 \qquad (k = 2, 3, \ldots, m) \end{aligned}$$

holds. We now suppose that the consistence order of the given method is p w.r.t. the model equation $y' = \lambda y$. Show that this implies $Q(e^{h\lambda}, h\lambda) = \mathcal{O}(h^{p+1})$ für $h \to 0$. Finally verify an estimate of the form $|Q(e^{h\lambda}, h\lambda)| \ge C|e^{h\lambda} - \xi_1(h\lambda)|$ with $|h\lambda|$ sufficiently small and an appropriate constant $C > 0$.

Hints to Exercise 8.8 The solution to the problem is similar to the proof of Theorem 8.57. For the determination of the general solution of the difference equation corresponding to the model equation, determine the roots $\lambda_{1/2}$ of the quadratic equation. For the further computations determine a Taylor expansion $\sqrt{1+t} = p_4(t) + \mathcal{O}(t^5)$ for $t \to 0$ with $p_4 \in \Pi_4$. Then show that expansions of the form $\lambda_1 = q_4(h) + \mathcal{O}(h^5)$ for $h \to 0$ and $\lambda_2 = q_1(h) + \mathcal{O}(h^2)$ for $h \to 0$ hold, with polynomials $q_4 \in \Pi_4$ and $q_1 \in \Pi_1$. From these expansions the coefficients α and β corresponding to the approximations $u_\ell = \alpha\lambda_1^\ell + \beta\lambda_2^\ell$ of the difference equation with initial values $u_0 = 1, u_1 = e^{-h}$ can be represented as follows: $\alpha = r_1(h) + \mathcal{O}(h^4)$ for $h \to 0$ and $\beta = r_4(h) + \mathcal{O}(h^5)$ for $h \to 0$, with polynomials $r_1 \in \Pi_1$ and $r_4 \in \Pi_4$. This finally leads to the required representation of u_ℓ. For the verification of the required asymptotical behavior use the identity $\ln(a+t) = \ln a + \mathcal{O}(t)$ for $t \to 0$.

Hints to Exercise 8.11 For quadratic matrices $A \in \mathbb{R}^{N\times N}$ we denote by $\mu_p[A]$ that logarithmic norm which is induced by the matrix norm $\|\cdot\|_p$. Verify the following representations:

$$\mu_\infty[A] \;=\; \max_{k=1..N}\Big(a_{kk} \;+\; \sum_{\substack{j=1\\ j\neq k}}^{N}|a_{kj}|\Big) \qquad \text{for} \;\; A \;=\; (a_{kj}) \in \mathbb{R}^{N\times N},$$

$$\mu_1[A] \;=\; \max_{j=1..N}\Big(a_{jj} \;+\; \sum_{\substack{k=1\\ k\neq j}}^{N}|a_{kj}|\Big) \qquad \text{———— « ————},$$

$$\mu_2[A] \;=\; \max\{\,\lambda \;:\; \lambda\in\sigma(A)\,\} \qquad \text{———— « ————}, \qquad A \;=\; A^\top.$$

Hints to Exercise 8.12 Using spacial central differences x leads to a system of ordinary differential equations for the approximations $y_k(t) \approx u(x_k,t),\; k = 0,1,\ldots,N$, with $x_k = k\Delta x$. In the corresponding differences use in the cases $k = 0$ respectively $k = N$ the settings $y_{-1} = y_1$ respectively $y_{N+1} = y_{N-1}$. These setting are justified by the Neumann−boundary conditions. The unknown logarithmic norm is induced by a well−known matrix norm.

Hints to Exercise 8.13 Show first that

$$\Big|\frac{\|e^{hA}\|-1}{h} \;-\; \frac{\|I+hA\|-1}{h}\Big| \to 0 \qquad \text{for} \;\; h\to 0.$$

For this an estimate the form $\sum_{k=j}^{\infty} x^k/k! \le e^x x^j$ is needed which also should be verified. From these considerations the statement in the hint to this exercise follows.

For the solution to this exercise show that

$$\Big|\frac{\ln\|e^{hA}\|}{h} \;-\; \frac{\|e^{hA}\|-1}{h}\Big| \;\to\; 0 \qquad \text{for} \;\; h\to 0$$

holds. For this use the estimate $|\ln(1+x)-x| \le x^2/(2\varepsilon^2)$ for $x \ge -1+\varepsilon$, where $0<\varepsilon\le 1$.

Hints to Exercise 8.14 For the verification of the triangle inequality use Exercise 8.13.

Hints to Exercise 8.15 Show first by an appropriate Taylor expansion that $(*)\;(1 \,+\, 2a\delta \,+\, b\delta^2)^{1/2} \;=\; 1 \,+\, a\delta \,+\, \mathcal{O}(\delta^2)$ for $\delta\to 0$ holds. This result should be applied several times in the sequel.

(a) Consider a normed eigenvector $x\in\mathbb{C}^N$ corresponding to the eigenvalue $\lambda\in\mathbb{C}$ of the matrix A and show that the identity $(\|(I+hA)x\|-1)/h = \mathbf{Re}\,\lambda \;+\; \mathcal{O}(h)$ for $h\to 0$ is satisfied. This finally yields the inequality stated in the exercise. Finally look for an example so that we have strict inequality.

(b) Consider first the number $\mu \;=\; \max_{\|x\|=1}\mathbf{Re}\,\langle Ax,x\rangle$ and show that $(\bullet)\;\mu[A] \;=\; \mu$ holds. For the proof of "$\ge$" in $(\bullet)$ consider a vector $x\in\mathbb{C}^N$ with $\|x\| = 1$ and $\mathbf{Re}\,\langle Ax,x\rangle \;=\; \mu$ and verify that $(\|(I+hA)x\| - 1)/h \;=\; \mu \;+\; \mathcal{O}(h)$ for $h\to 0$ holds. As a preparation for the proof of the estimate "$\le$" in $(\bullet)$ show that $(\|(I+hA)x\|-1)/h \le \mu \;+\; \mathcal{O}(h)$ for $h\to 0$ holds for each vector $x\in\mathbb{C}^N$ with $\|x\| = 1$, where $\mathcal{O}(h)$ is independent of x.

Hints to Exercise 8.16 For the verification of "$\Longrightarrow$" one has to show first that the estimates

$$|1 \,+\, \mu a_{kk}| \;+\; \mu\sum_{j\neq k}|a_{kj}| \;\le\; 1 \qquad \text{for} \;\; k \;=\; 1,2,\ldots,N$$

are satisfied for each number μ with $0 \le \mu\|A\|_\infty \le 2$. For this purpose consider the two cases "$1 \le \mu|a_{kk}|$" and "$0 \le \mu|a_{kk}| \le 1$" separately. In the second case use the hint for Exercise 8.11 with the explicit representation for $\mu_\infty[A]$. The reverse implication "$\Longleftarrow$" is trivial.