The Inverse Problem for Kreĭn Orthogonal Entire Matrix Functions

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joint work with I. Gohberg and L. Lerer

Let $k \in L_1^{n\times n}[-\omega,\omega]$, and let k be hermitian, that is, $k(t)^* = k(-t)$ for $-\omega \le t \le \omega$. Assume the equation

$$\varphi(t) - \int_0^\omega k(t-s)\varphi(s)\,ds = k(t), \quad 0 \le t \le \omega, \tag{1}$$

has a solution $\varphi \in L_1^{n\times n}[0,\omega]$, and consider the associated entire $n\times n$ matrix function

$$\Phi(\lambda) = I + \int_0^\omega e^{i\lambda t}\varphi(t)\,dt, \quad \lambda \in \mathbb{C}. \tag{2}$$

For the scalar case ($n = 1$) functions of type (2) determined by (1) have been introduced by M.G. Kreĭn as continuous analogues of the classical Szegö orthogonal polynomials with respect to the unit circle. For this reason Φ is referred to as a *Kreĭn orthogonal function* generated by k. Kreĭn in the fifties and Kreĭn and Langer in the eighties proved a number of remarkable results (still for $n = 1$). One of these results is the solution of the inverse problem: For a scalar function Φ of the form (2) there exists a hermitian $k \in L_1[-\omega,\omega]$ such that (1) holds if and only if Φ has no real zeroes and no conjugate pairs of zeroes. Many of the Kreĭn-Langer results have been extended to matrix-valued functions. The following theorem, which seems to be new, gives the solution to the inverse problem for matrix-valued functions.

Theorem 1 *For a function Φ defined by* (2) *there exists a hermitian k in $L_1^{n\times n}[-\omega,\omega]$ such that* (1) *holds if and only if $\det\Phi(\lambda)$ has no real zero and for any symmetric pair of zeros $\lambda_0, \bar\lambda_0$ of $\det\Phi(\lambda)$ we have*

$$\sum_{j=0}^{k} \langle \varphi_{k-j}, \psi_j \rangle_{\mathbb{C}^n} = 0, \quad k = 0,1,\ldots,\min\{p,q\}-1, \tag{3}$$

where $\varphi_0, \varphi_1, \ldots, \varphi_{p-1}$ *and* $\psi_0, \psi_1, \ldots, \psi_{q-1}$ *are arbitrary Jordan chains for* Φ *at* λ_0 *and* $\bar{\lambda}_0$*, respectively.*

The proof is more involved than the one for the scalar counterpart. New techniques based on recent results from the theory of continuous analogues of resultant and Bezout matrices are required. Also solutions to certain equations in entire matrix functions enter into the proof. In the talk the various steps in the proof will be reviewed.