The Abstract Kinetic Equation on a Half-Line

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Consider the boundary value problem

$$w(\mu)\frac{\partial\psi}{\partial x}(x,\mu) = \frac{\partial^2\psi}{\partial\mu^2}(x,\mu) - q(\mu)\psi(x,\mu) \qquad (0 < x < \infty,\ \mu \in \mathbb{R}), \tag{1}$$

$$\psi(0,\mu) = \phi_+(\mu) \text{ if } w(\mu) > 0, \qquad \int_{\mathbb{R}} |\psi(x,\mu)|^2 |w(\mu)| d\mu = O(1) \text{ as } x \to \infty \tag{2}$$

Here the weight function w changes its sign on $\mathbb{R}$. We assume that the operator $L : y \mapsto |w|^{-1}(-y'' + qy)$ is a self-adjoint operator in the Hilbert space $L^2(\mathbb{R}, |w(x)|dx)$. Boundary value problems of this type arise as various kinetic equations (e.g. [1, 2] and references).

We consider the abstract kinetic equation in the following form:

$$\frac{d\psi}{dx} = -JL\psi(x) \qquad (0 < x < \infty), \tag{3}$$

where J and L are operators in the abstract Hilbert space H such that $J = J^{-1} = J^*$ is an *signature operator* in H and L is a self-adjoint operator. By $P_\pm$ we denote the orthogonal projections onto $H_\pm := \ker(J \mp I)$. The problem is to find continuous functions $\psi : [0, +\infty) \to H$ which is H-differentiable on $(0, \infty)$ and satisfies Eq. (3) with the following boundary conditions

$$P_+\psi(0) = \phi_+, \tag{4}$$

$$\|\psi(x)\|_H = O(1) \qquad (x \to +\infty), \tag{5}$$

where $\phi_+ \in H_+$ is a given vector.

The case when L is nonnegative and has discrete spectrum has been described in great detail in papers of Beals, Kaper, Protopopescu, van der Mee, Pyatkov and other authors.

First we eliminate the assumption that the spectrum of L is discrete.

Theorem 1 *Assume that* $L = L^* \geq 0$, $\ker L = 0$, *and* J*-nonnegative operator* $A := JL$ *is definitizable. Assume that neither* 0 *nor* ∞ *are singular critical points of* A. *Then for each* $\phi_+ \in H_+$, *there is a unique solution* ψ *of* (3)-(5).

Secondly, the case when the negative spectrum of L is nonempty is considered.

Let us introduce the following boundary condition at ∞

$$(E^\alpha): \qquad \|\psi(x)\|_H = O(e^{\alpha x}) \qquad (x \to +\infty).$$

Theorem 2 *Suppose that* L *is bounded below, that* $A := JL$ *is a definitizable operator, and* ∞ *is not a singular critical point of* A. *Then there exist* $\alpha \in \mathbb{R}$ *such that problem* (3)-(4)-(E^α) *has solutions for each* $\phi_+ \in H_+$.

References

[1] *van der Mee, C.V.M., Ran, A.C.M., Rodman, L.* Stability of stationary transport equation with accretive collision operators. J. Funct. Anal. **174** (2000), 478–512.

[2] *Pyatkov, S.G.* Operator Theory. Nonclassical Problems. Utrecht, Boston, Köln, Tokyo, VSP 2002.