Eigenvalue Problems with Boundary Conditions Depending Polynomially on the Eigenparameter

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Let S be a closed densely defined symmetric operator with equal defect numbers $d < \infty$ in a Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot \rangle)$. Let $\mathsf{b} : \mathrm{dom}(S^*) \to \mathbb{C}^{2d}$ be a boundary mapping for S. We assume that S has a self-adjoint extension with a compact resolvent. Let $\mathcal{P}(z)$ be a $d \times 2d$ matrix polynomial.

We will give sufficient conditions on $\mathcal{P}(z)$ under which the eigenvalue problem

$$S^* f = \lambda f, \quad \mathcal{P}(\lambda)\mathsf{b}(f) = 0$$

is equivalent to an eigenvalue problem for a self-adjoint operator $\widetilde{A}$ in a Pontrjagin space which is the direct sum of $\mathcal{H}$ and a finite-dimensional space. Both, this finite dimensional Pontrjagin space and the self-adjoint operator $\widetilde{A}$ are defined explicitly in terms of the coefficients of $\mathcal{P}(z)$.

In a special case when S is associated with an ordinary regular differential expression we give a description of the form domain of the operator $\widetilde{A}$ in terms of the essential boundary conditions. It is shown that the eigenfunction expansions for the elements in the form domain converge in a topology that is stronger than uniform.

If J is a self-adjoint involution on $\mathcal{H}$ and JS has a definitizable extension in the Krein space $(\mathcal{H}, \langle J\,\cdot\,, \cdot \rangle)$ our results extend to the eigenvalue problem in which S^* is replaced by JS^*.

As a model problem we propose the following

$$-f''(x) = \lambda(\operatorname{sgn} x) f(x), \quad x \in [-1, 1],$$

$$\begin{pmatrix} 1 & 0 & 0 & \lambda \\ 0 & 1 & -\lambda & \lambda^n \end{pmatrix} \begin{pmatrix} f(-1) \\ f(1) \\ f'(-1) \\ f'(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$