Forward-Backward Kinetic Equations and the Similarity Problem for Sturm-Liouville Operators

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Consider the equation $r(v)\psi_x(x,v) = \psi_{vv}(x,v) - q(v)\psi(x,v) + f(x,v)$, $0 < x < 1$, $v \in \mathbb{R}$, and the associated half-range boundary value problem $\psi(0,v) = \varphi_+(v)$ if $v > 0$, $\psi(1,v) = \varphi_-(v)$ if $v < 0$. It is assumed that $vr(v) > 0$. So the weight function r changes its sign at 0. Boundary value problems of this type arise as various kinetic equations.

We consider the above equation in the abstract form $J\psi_x(x) + L\psi(x) = f(x)$, where J and L are operators in a Hilbert space H such that $J = J^* = J^{-1}$, $L = L^* \geq 0$, and $\ker L = 0$. The case when L is nonnegative and has discrete spectrum or satisfies the weaker assumption $\inf \sigma_{ess}(L) > 0$ was described in great detail (see [4] and references therein). The latter assumption is not fulfilled for some physical models. The simplest example is the equation

$$v\psi_x(x,v) = \psi_{vv}(x,v), \quad 0 < x < 1, \quad v \in \mathbb{R}, \tag{1}$$

which was studied in a number of papers during last 50 years. The complete existence and uniqueness theory for equations of such type have not been constructed.

It will be shown that the method of [1] can be modified to prove the following theorem: *if the J-self-adjoint operator JL is similar to a self-adjoint one, then the associated half-range boundary problem has a unique solution for arbitrary $\varphi_\pm \in L^2(\mathbb{R}_\pm, |r|)$.* The latter can be applied to (1) due to the result of Fleige and Najman on the similarity of the operator $(\operatorname{sgn} v)|v|^{-\alpha}\frac{d^2}{dv^2}$, $\alpha > -1$. Connections between equations of type (1) and the recent papers [3, 2] will be considered also.

References

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