Bissectors and Isometries on Hilbert Spaces

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Let $\mathcal{H}$ be a Hilbert space over $\mathbf{C}$ and let $F(\mathcal{H})$ be the set of all closed linear subspaces of $\mathcal{H}$. For all $M,\, N \in F(\mathcal{H})$ set $g(M, N) = \|P_M - P_N\|$ (known as the *gap metric*) where P_M, P_N denote respectively the orthogonal projections in $\mathcal{H}$ on M and on N.

For all $M,\, N \in F(\mathcal{H})$ such that $ker(P_M + P_N - I) = \{0\}$, $\Psi(M, N)$, the *bissector* of M and N, is a uniquely determined element of $F(\mathcal{H})$ such that (setting $\Psi(M, N) = W$):

(i) $P_M P_W = P_W P_N$

(ii) $(P_M + P_N)P_W = P_W(P_M + P_N)$ is positive definite.

A mapping Φ of $F(\mathcal{H})$ into itself is called an *isometry* if

$$\forall M,\, N \in F(\mathcal{H}),\, g(M, N) = g(\Phi(M), \Phi(N)).$$

Theorem : Let $M,\, N \in F(\mathcal{H})$ be such that $ker(P_M + P_N - I) = \{0\}$ and let Φ be an isometry on $F(\mathcal{H})$. Then if $ker(P_{\Phi(M)} + P_{\Phi(N)} - I) = \{0\}$:

$$\Phi(\Psi(M, N)) = \Psi(\Phi(M), \Phi(N)).$$

A number of applications of this result are given.