Inertia Theorems Based on Operator Lyapunov Equations

A.C.M. Ran
joint work with L. Lerer and I. Margulis

In 1962 D. Carlson and H. Schneider proved the following result. For an $n \times n$ complex matrix A let $\pi(A), \nu(A)$ and $\delta(A)$ denote the number of eigenvalues, counting multiplicities, located in the right halfplane, the left halfplane, and on the imaginary axis, respectively.

Theorem 1 *Let $A \in \mathbb{C}^{n \times n}$ and let X be a Hermitian matrix such that*

$$AX + XA^* = W \geq 0.$$

(i) If $\delta(A) = 0$, then $\pi(X) \leq \pi(A)$ and $\nu(X) \leq \nu(A)$.

(ii) If X is nonsingular, then $\pi(A) \leq \pi(X)$ and $\nu(A) \leq \nu(X)$.

(iii) From (i) and (ii) it follows that if $\delta(A) = \delta(X) = 0$, then $\pi(X) = \pi(A)$ and $\nu(X) = \nu(A)$.

The main goal of the lecture is to extend the third part of this result to the case of possibly unbounded linear operators acting on infinite dimensional Hilbert spaces. The second part was already generalized in earlier work of Curtain and Sasane. The following theorem is one of the main results.

Theorem 2 *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear, densely defined closed operator with domain $D(A)$. Suppose, $H \in L(\mathcal{H})$ is a self-adjoint invertible operator such that $\nu(H) < \infty$ and*

$$\langle (A^*H + HA)x, x \rangle \geq 0, \qquad \forall x \in D(A). \tag{1}$$

Assume, in addition, that A is boundedly invertible, the spectrum of A does not contain eigenvalues which lie on the imaginary axis, and $\sigma(A) \cap \mathbb{C}^-$ is a bounded spectral set. Then $\nu(H) = \nu(A)$.

The proof of the theorem makes use of the theory of operators in indefinite inner product spaces.