Non-negative perturbations of non-negative selfadjoint operators

Vadim Adamyan

Department of Theoretical Physics
Odessa National University

Berlin, December 14 -17, 2006

Problem motivation and definition

We consider

- Laplace operators $-\Delta$ in $L_2(\mathbf{R}_3)$ and $L_2(\mathbf{R}_2)$;
- the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H_2^2(\mathbf{R}_l \setminus \{0\})$;
- self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$ of $-\Delta^0$ in $L_2(\mathbf{R}_l)$, $l = 3, 2$.

Problem motivation and definition

We consider

- Laplace operators $-\Delta$ in $L_2(\mathbf{R}_3)$ and $L_2(\mathbf{R}_2)$;
- the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H_2^2(\mathbf{R}_l \setminus \{0\})$;
- self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$ of $-\Delta^0$ in $L_2(\mathbf{R}_l)$, $l = 3, 2$.

Problem motivation and definition

We consider

- Laplace operators $-\Delta$ in $L_2(\mathbf{R}_3)$ and $L_2(\mathbf{R}_2)$;
- the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H_2^2(\mathbf{R}_i \setminus \{0\})$;
- self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$ of $-\Delta^0$ in $L_2(\mathbf{R}_i)$, $i = 3, 2$.

Problem motivation and definition

We consider

- Laplace operators $-\Delta$ in $L_2(\mathbf{R}_3)$ and $L_2(\mathbf{R}_2)$;
- the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H_2^2(\mathbf{R}_i \setminus \{0\})$;
- self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$ of $-\Delta^0$ in $L_2(\mathbf{R}_i)$, $i = 3, 2$.

Problem motivation and definition

We consider

- Laplace operators $-\Delta$ in $L_2(\mathbf{R}_3)$ and $L_2(\mathbf{R}_2)$;
- the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H_2^2(\mathbf{R}_i \setminus \{0\})$;
- self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$ of $-\Delta^0$ in $L_2(\mathbf{R}_i)$, $i = 3, 2$.

Problem motivation and definition

We consider

- Laplace operators $-\Delta$ in $L_2(\mathbf{R}_3)$ and $L_2(\mathbf{R}_2)$;
- the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H_2^2(\mathbf{R}_i \setminus \{0\})$;
- self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$ of $-\Delta^0$ in $L_2(\mathbf{R}_i)$, $i = 3, 2$.

Problem motivation and definition

Domains of $-\Delta_\alpha$:

$$\mathcal{D}_\alpha^{(3)} := \left\{ f : \ f \in H_2^2(\mathbf{R}_3), \lim_{|\mathbf{x}|\downarrow 0} \left[\frac{d}{d|\mathbf{x}|} (|\mathbf{x}| f(\mathbf{x})) - \alpha |\mathbf{x}| f(\mathbf{x}) \right] = 0 \right\},$$

$$\mathcal{D}_\alpha^{(2)} := \left\{ f : \ f \in H_2^2(\mathbf{R}_2), \lim_{|\mathbf{x}|\downarrow 0} \left[\left(\frac{2\pi\alpha}{\ln|\mathbf{x}|} + 1 \right) f(\mathbf{x}) - \lim_{|\mathbf{x}'|\downarrow 0} \frac{\ln|\mathbf{x}|}{\ln|\mathbf{x}'|} f(\mathbf{x}') \right] = 0. \right\}$$

Domains of $-\Delta_\alpha$:

$$\mathcal{D}_\alpha^{(3)} := \left\{ f : f \in H_2^2(\mathbf{R}_3), \lim_{|\mathbf{x}|\downarrow 0} \left[\frac{d}{d|\mathbf{x}|} (|\mathbf{x}| f(\mathbf{x})) - \alpha |\mathbf{x}| f(\mathbf{x}) \right] = 0 \right\},$$

$$\mathcal{D}_\alpha^{(2)} := \left\{ f : f \in H_2^2(\mathbf{R}_2), \lim_{|\mathbf{x}|\downarrow 0} \left[\left(\frac{2\pi\alpha}{\ln |\mathbf{x}|} + 1 \right) f(\mathbf{x}) - \lim_{|\mathbf{x}'|\downarrow 0} \frac{\ln |\mathbf{x}|}{\ln |\mathbf{x}'|} f(\mathbf{x}') \right] = 0. \right\}$$

Problem motivation and definition

Domains of $-\Delta_\alpha$:

$$\mathcal{D}_\alpha^{(3)} := \left\{ f : \ f \in H_2^2(\mathbf{R}_3), \lim_{|\mathbf{x}|\downarrow 0} \left[\frac{d}{d|\mathbf{x}|}(|\mathbf{x}| f(\mathbf{x})) - \alpha |\mathbf{x}| f(\mathbf{x}) \right] = 0 \right\},$$

$$\mathcal{D}_\alpha^{(2)} := \left\{ f : \ f \in H_2^2(\mathbf{R}_2), \lim_{|\mathbf{x}|\downarrow 0} \left[\left(\frac{2\pi\alpha}{\ln|\mathbf{x}|} + 1 \right) f(\mathbf{x}) - \lim_{|\mathbf{x}'|\downarrow 0} \frac{\ln|\mathbf{x}|}{\ln|\mathbf{x}'|} f(\mathbf{x}') \right] = 0. \right\}$$

Problem motivation and definition

Resolvent kernels (Green functions):

$$G^{(3)}_{\alpha,z}(\mathbf{x},\mathbf{x}') = \begin{cases} G^{(0)}_z(\mathbf{x},\mathbf{x}') + \frac{1}{\alpha - i\sqrt{z}/4\pi}\, G^{(0)}_z(\mathbf{x},0)G^{(0)}_z(0,\mathbf{x}')\,, \\ \\ G^{(0)}_z(\mathbf{x},\mathbf{x}') = \frac{\exp i\sqrt{z}|\mathbf{x}-\mathbf{x}'|}{4\pi|\mathbf{x}-\mathbf{x}'|}\,. \end{cases}$$

$$G^{(2)}_{\alpha,z}(\mathbf{x},\mathbf{x}') = \begin{cases} G^{(0)}_z(\mathbf{x},\mathbf{x}') + \frac{2\pi}{2\pi\alpha - \psi(1) + \ln\left(\frac{\sqrt{z}}{2i}\right)}\, G^{(0)}_z(\mathbf{x},0)G^{(0)}_z(0,\mathbf{x}')\,, \\ \\ G^{(0)}_z(\mathbf{x},\mathbf{x}') = \left(\frac{i}{4}\right)H^{(1)}_0(i\sqrt{z}|\mathbf{x}-\mathbf{x}'|)\,. \end{cases}$$

Problem motivation and definition

Resolvent kernels (Green functions):

$$G^{(3)}_{\alpha,z}(\mathbf{x},\mathbf{x}') = \begin{cases} G^{(0)}_z(\mathbf{x},\mathbf{x}') + \frac{1}{\alpha - i\sqrt{z}/4\pi}\, G^{(0)}_z(\mathbf{x},0)G^{(0)}_z(0,\mathbf{x}')\,, \\ \\ G^{(0)}_z(\mathbf{x},\mathbf{x}') = \frac{\exp i\sqrt{z}|\mathbf{x}-\mathbf{x}'|}{4\pi|\mathbf{x}-\mathbf{x}'|}\,. \end{cases}$$

$$G^{(2)}_{\alpha,z}(\mathbf{x},\mathbf{x}') = \begin{cases} G^{(0)}_z(\mathbf{x},\mathbf{x}') + \frac{2\pi}{2\pi\alpha - \psi(1) + \ln\left(\frac{\sqrt{z}}{2i}\right)}\, G^{(0)}_z(\mathbf{x},0)G^{(0)}_z(0,\mathbf{x}')\,, \\ \\ G^{(0)}_z(\mathbf{x},\mathbf{x}') = \left(\frac{i}{4}\right)H^{(1)}_0(i\sqrt{z}|\mathbf{x}-\mathbf{x}'|)\,. \end{cases}$$

Problem motivation and definition

Resolvent kernels (Green functions):

$$G^{(3)}_{\alpha,z}(\mathbf{x},\mathbf{x}') = \begin{cases} G^{(0)}_z(\mathbf{x},\mathbf{x}') + \frac{1}{\alpha - i\sqrt{z}/4\pi}\, G^{(0)}_z(\mathbf{x},0) G^{(0)}_z(0,\mathbf{x}')\,, \\ \\ G^{(0)}_z(\mathbf{x},\mathbf{x}') = \frac{\exp i\sqrt{z}|\mathbf{x}-\mathbf{x}'|}{4\pi|\mathbf{x}-\mathbf{x}'|}\,. \end{cases}$$

$$G^{(2)}_{\alpha,z}(\mathbf{x},\mathbf{x}') = \begin{cases} G^{(0)}_z(\mathbf{x},\mathbf{x}') + \frac{2\pi}{2\pi\alpha - \psi(1) + \ln\left(\frac{\sqrt{z}}{2i}\right)}\, G^{(0)}_z(\mathbf{x},0) G^{(0)}_z(0,\mathbf{x}')\,, \\ \\ G^{(0)}_z(\mathbf{x},\mathbf{x}') = \left(\frac{i}{4}\right) H^{(1)}_0(i\sqrt{z}|\mathbf{x}-\mathbf{x}'|)\,. \end{cases}$$

Problem motivation and definition

All singular perturbations $-\Delta_\alpha$ of the Laplace operator in two dimensions have one negative eigenvalue or the standardly defined Laplace operator $-\Delta$ is the unique non-negative self-adjoint extension in $L_2(\mathbf{R}_2)$ of the symmetric operator $-\Delta^0$.

Problem motivation and definition: Question

Question

Why in some cases the Friedrichs extension is the unique non-negative extension of given non-negative symmetric operator?

Problem motivation and definition: Question

Question

Why in some cases the Friedrichs extension is the unique non-negative extension of given non-negative symmetric operator?

- $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$
- $A^{(0)}$ be a densely defined closed restriction of A onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of A.

Put

$$\mathcal{M} := (I + A^{(0))}\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$
$$\mathcal{N} := \mathcal{H} \ominus \mathcal{M}.$$

We call all self-adjoiont extensions of $A^{(0)}$ in $\mathcal{H}$ other than A *singular perturbations* of A (associated with A_0).

- $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$
- $A^{(0)}$ be a densely defined closed restriction of A onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of A.

Put

$$\mathcal{M} := (I + A^{(0)})\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$
$$\mathcal{N} := \mathcal{H} \ominus \mathcal{M}.$$

We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than A *singular perturbations* of A (associated with A_0).

- $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$
- $A^{(0)}$ be a densely defined closed restriction of A onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of A.

Put

$$\mathcal{M} := (I + A^{(0)})\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$
$$\mathcal{N} := \mathcal{H} \ominus \mathcal{M}.$$

We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than A *singular perturbations* of A (associated with A_0).

- $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$
- $A^{(0)}$ be a densely defined closed restriction of A onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of A.

Put

$$\mathcal{M} := (I + A^{(0))}\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$
$$\mathcal{N} := \mathcal{H} \ominus \mathcal{M}.$$

We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than A *singular perturbations* of A (associated with A_0).

- $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$
- $A^{(0)}$ be a densely defined closed restriction of A onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of A.

Put

$$\mathcal{M} := (I + A^{(0))}\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$
$$\mathcal{N} := \mathcal{H} \ominus \mathcal{M}.$$

We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than A *singular perturbations* of A (associated with A_0).

- $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$
- $A^{(0)}$ be a densely defined closed restriction of A onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of A.

Put

$$\mathcal{M} := (I + A^{(0))}\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$
$$\mathcal{N} := \mathcal{H} \ominus \mathcal{M}.$$

We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than A *singular perturbations* of A (associated with A_0).

- $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$
- $A^{(0)}$ be a densely defined closed restriction of A onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of A.

Put

$$\mathcal{M} := (I + A^{(0)})\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$
$$\mathcal{N} := \mathcal{H} \ominus \mathcal{M}.$$

We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than A *singular perturbations* of A (associated with A_0).

Let us consider $K_0 : \mathcal{M} \to \mathcal{H}$:

$$\begin{cases} f = \left(I + A^{(0)}\right) x, \\ K_0 f = \mathcal{A}^{(0)} x, \ x \in \mathcal{D}(A^{(0)}). \end{cases}$$

A_1 is a non-negative self-adjoint extension of A_0 in $\mathcal{H}$ iff $K_1 := A_1 \left(A_1 + I\right)^{-1}$ is a non-negative contractive extension of K_0 from $\mathcal{M}$ onto $\mathcal{H}$, $K_1 f = K_0 f$, $f \in \mathcal{M}, 1 \underline{\in} \sigma(K_1)$.

A_0 has unique non-negative self-adjoint extension in $\mathcal{H}$ if and only if K_0 admits only one non-negative contractive extension onto the whole $\mathcal{H}$, no eigenvalue of which $= 1$, that is $K := A(I + A)^{-1}$.

The uniqueness of A as non-negative extension of A_0 is equivalent to uniqueness of K as non-negative contractive extension of K_0.

Let us consider $K_0 : \mathcal{M} \to \mathcal{H}$:

$$\begin{cases} f = \left(I + A^{(0)}\right) x, \\ K_0 f = \mathcal{A}^{(0)} x, \; x \in \mathcal{D}(A^{(0)}). \end{cases}$$

A_1 is a non-negative self-adjoint extension of A_0 in $\mathcal{H}$ iff $K_1 := A_1 \left(A_1 + I\right)^{-1}$ is a non-negative contractive extension of K_0 from $\mathcal{M}$ onto $\mathcal{H}$, $K_1 f = K_0 f, \; f \in \mathcal{M}, 1 \underline{\in} \sigma(K_1)$.

A_0 has unique non-negative self-adjoint extension in $\mathcal{H}$ if and only if K_0 admits only one non-negative contractive extension onto the whole $\mathcal{H}$, no eigenvalue of which $= 1$, that is $K := A(I + A)^{-1}$.

The uniqueness of A as non-negative extension of A_0 is equivalent to uniqueness of K as non-negative contractive extension of K_0.

Let us consider $K_0 : \mathcal{M} \to \mathcal{H}$:

$$\begin{cases} f = \left(I + A^{(0)}\right) x, \\ K_0 f = \mathcal{A}^{(0)} x, \ x \in \mathcal{D}(A^{(0)}). \end{cases}$$

A_1 is a non-negative self-adjoint extension of A_0 in $\mathcal{H}$ iff $K_1 := A_1 (A_1 + I)^{-1}$ is a non-negative contractive extension of K_0 from $\mathcal{M}$ onto $\mathcal{H}$, $K_1 f = K_0 f,\ f \in \mathcal{M}, 1 \underline{\in} \sigma(K_1)$.

A_0 has unique non-negative self-adjoint extension in $\mathcal{H}$ if and only if K_0 admits only one non-negative contractive extension onto the whole $\mathcal{H}$, no eigenvalue of which $= 1$, that is $K := A(I + A)^{-1}$.

The uniqueness of A as non-negative extension of A_0 is equivalent to uniqueness of K as non-negative contractive extension of K_0.

Let us consider $K_0 : \mathcal{M} \to \mathcal{H}$:

$$\begin{cases} f = \left(I + A^{(0)}\right) x, \\ K_0 f = \mathcal{A}^{(0)} x,\ x \in \mathcal{D}(A^{(0)}). \end{cases}$$

A_1 is a non-negative self-adjoint extension of A_0 in $\mathcal{H}$ iff $K_1 := A_1 \left(A_1 + I\right)^{-1}$ is a non-negative contractive extension of K_0 from $\mathcal{M}$ onto $\mathcal{H}$, $K_1 f = K_0 f$, $f \in \mathcal{M}, 1 \underline{\in} \sigma(K_1)$.

A_0 has unique non-negative self-adjoint extension in $\mathcal{H}$ if and only if K_0 admits only one non-negative contractive extension onto the whole $\mathcal{H}$, no eigenvalue of which $= 1$, that is $K := A(I + A)^{-1}$.

The uniqueness of A as non-negative extension of A_0 is equivalent to uniqueness of K as non-negative contractive extension of K_0.

Notation

Notation

- **G** - the set consisting of A and all its non-negative singular perturbations;
- **C** denote the set of non-negative contractions obtained from **G** by transformation $A_1 \to A_1\left(A_1+I\right)^{-1}$, $A_1 \in$ **G**;
- $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$
- $P_{\mathcal{N}} = I - P_{\mathcal{M}}$.

Notation

- **G** - the set consisting of A and all its non-negative singular perturbations;
- **C** denote the set of non-negative contractions obtained from **G** by transformation $A_1 \to A_1 (A_1 + I)^{-1}$, $A_1 \in$ **G**;
- $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$
- $P_{\mathcal{N}} = I - P_{\mathcal{M}}$.

Notation

Notation

- **G** - the set consisting of A and all its non-negative singular perturbations;
- **C** denote the set of non-negative contractions obtained from **G** by transformation $A_1 \to A_1 (A_1 + I)^{-1}$, $A_1 \in$ **G**;
- $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$
- $P_{\mathcal{N}} = I - P_{\mathcal{M}}$.

Notation

Notation

- **G** - the set consisting of A and all its non-negative singular perturbations;
- **C** denote the set of non-negative contractions obtained from **G** by transformation $A_1 \to A_1 (A_1 + I)^{-1}$, $A_1 \in$ **G**;
- $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$
- $P_{\mathcal{N}} = I - P_{\mathcal{M}}$.

Notation

Notation

- **G** - the set consisting of A and all its non-negative singular perturbations;
- **C** denote the set of non-negative contractions obtained from **G** by transformation $A_1 \rightarrow A_1 \left(A_1 + I\right)^{-1}$, $A_1 \in$ **G**;
- $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$
- $P_{\mathcal{N}} = I - P_{\mathcal{M}}$.

With respect to the representation $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ each $K_X \in$ **C** can be represented as

$$K_X = \begin{pmatrix} T & \Gamma^* \\ \Gamma & X \end{pmatrix}$$

Here

$$T = P_{\mathcal{M}} K_0|_{\mathcal{M}},$$
$$\Gamma = P_{\mathcal{M}} K_0|_{\mathcal{M}}.$$

X is a non-negative contraction in $\mathcal{N}$ distinguishing elements from **C**.

Since each $K_X \in \mathbf{C}$ is non-negative and contractive then

$$T \geq 0; \quad I \geq T^2 + \Gamma^*\Gamma$$

$K_X \in \mathbf{C}$ is equivalent to

$$K_X + \varepsilon I \geq 0;$$
$$(1+\varepsilon)I - K_X \geq 0 \ \varepsilon > 0.$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ \Gamma(T+\varepsilon)^{-1} & I \end{pmatrix} \times \begin{pmatrix} T+\varepsilon & 0 \\ 0 & X+\varepsilon-\Gamma(T+\varepsilon)^{-1}\Gamma^* \end{pmatrix} \times \begin{pmatrix} I & (T+\varepsilon)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ \Gamma(T+\varepsilon)^{-1} & I \end{pmatrix} \times$$
$$\begin{pmatrix} T+\varepsilon & 0 \\ 0 & X+\varepsilon-\Gamma(T+\varepsilon)^{-1}\Gamma^* \end{pmatrix} \times$$
$$\begin{pmatrix} I & (T+\varepsilon)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ \Gamma(T+\varepsilon)^{-1} & I \end{pmatrix} \times$$

$$\begin{pmatrix} T+\varepsilon & 0 \\ 0 & X+\varepsilon-\Gamma(T+\varepsilon)^{-1}\Gamma^* \end{pmatrix} \times$$

$$\begin{pmatrix} I & (T+\varepsilon)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ \Gamma(T+\varepsilon)^{-1} & I \end{pmatrix} \times \begin{pmatrix} T+\varepsilon & 0 \\ 0 & X+\varepsilon-\Gamma(T+\varepsilon)^{-1}\Gamma^* \end{pmatrix} \times \begin{pmatrix} I & (T+\varepsilon)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ -\Gamma(I+\varepsilon-T)^{-1} & I \end{pmatrix} \times$$
$$\begin{pmatrix} 1+\varepsilon-T & 0 \\ 0 & 1+\varepsilon-X-\Gamma(1+\varepsilon-T)^{-1}\Gamma^{*} \end{pmatrix} \times$$
$$\begin{pmatrix} I & -(1+\varepsilon-T)^{-1}\Gamma^{*} \\ 0 & I \end{pmatrix} \geq 0$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ -\Gamma(I+\varepsilon-T)^{-1} & I \end{pmatrix} \times$$

$$\begin{pmatrix} 1+\varepsilon-T & 0 \\ 0 & 1+\varepsilon-X-\Gamma(1+\varepsilon-T)^{-1}\Gamma^* \end{pmatrix} \times$$

$$\begin{pmatrix} I & -(1+\varepsilon-T)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ -\Gamma(I+\varepsilon-T)^{-1} & I \end{pmatrix} \times$$

$$\begin{pmatrix} 1+\varepsilon-T & 0 \\ 0 & 1+\varepsilon-X-\Gamma(1+\varepsilon-T)^{-1}\Gamma^* \end{pmatrix} \times$$

$$\begin{pmatrix} I & -(1+\varepsilon-T)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0$$

By the Schur -Frobenius factorization formula:

$$\begin{pmatrix} I & 0 \\ -\Gamma(I+\varepsilon-T)^{-1} & I \end{pmatrix} \times \begin{pmatrix} 1+\varepsilon-T & 0 \\ 0 & 1+\varepsilon-X-\Gamma(1+\varepsilon-T)^{-1}\Gamma^* \end{pmatrix} \times \begin{pmatrix} I & -(1+\varepsilon-T)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0$$

Since $T \geq 0$ and $I - T \geq 0$ the above inequalities are reduced to

$$\begin{cases} X + \varepsilon I - \Gamma(T + \varepsilon I)^{-1}\Gamma^* \geq 0, \\ (1+\varepsilon)I - X - \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \geq 0, \ \varepsilon > 0. \end{cases}$$

Setting

$$Y := X - \lim_{\varepsilon \downarrow 0} \Gamma(T + \varepsilon I)^{-1}\Gamma^*$$

we conclude that $K_X \in \mathbf{C}$ if and only if

$$0 \leq Y \leq I - \lim_{\varepsilon \downarrow 0} \left(\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right).$$

Since $T \geq 0$ and $I - T \geq 0$ the above inequalities are reduced to

$$\begin{cases} X + \varepsilon I - \Gamma(T + \varepsilon I)^{-1}\Gamma^* \geq 0, \\ (1+\varepsilon)I - X - \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \geq 0, \ \varepsilon > 0. \end{cases}$$

Setting

$$Y := X - \lim_{\varepsilon \downarrow 0} \Gamma(T + \varepsilon I)^{-1}\Gamma^*$$

we conclude that $K_X \in \mathbf{C}$ if and only if

$$0 \leq Y \leq I - \lim_{\varepsilon \downarrow 0} \left(\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right).$$

Since $T \geq 0$ and $I - T \geq 0$ the above inequalities are reduced to

$$\begin{cases} X + \varepsilon I - \Gamma(T + \varepsilon I)^{-1}\Gamma^* \geq 0, \\ (1+\varepsilon)I - X - \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \geq 0, \ \varepsilon > 0. \end{cases}$$

Setting

$$Y := X - \lim_{\varepsilon \downarrow 0} \Gamma(T + \varepsilon I)^{-1}\Gamma^*$$

we conclude that $K_X \in \mathbf{C}$ if and only if

$$0 \leq Y \leq I - \lim_{\varepsilon \downarrow 0} \left(\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right).$$

Hence

$$I - \lim_{\varepsilon \downarrow 0} \left(\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right) = 0$$

is the criterium that there are no contractive non-negative extension of K_0 in $\mathcal{H}$ other than K.

To express this criterium in terms of given K and A we use the following proposition.

Hence

$$I - \lim_{\varepsilon \downarrow 0} \left(\Gamma (T + \varepsilon I)^{-1} \Gamma^* + \Gamma [(1 + \varepsilon) I - T]^{-1} \Gamma^* \right) = 0$$

is the criterium that there are no contractive non-negative extension of K_0 in $\mathcal{H}$ other than K.

To express this criterium in terms of given K and A we use the following proposition.

Hence

$$I - \lim_{\varepsilon \downarrow 0} \left(\Gamma (T + \varepsilon I)^{-1} \Gamma^* + \Gamma [(1+\varepsilon) I - T]^{-1} \Gamma^* \right) = 0$$

is the criterium that there are no contractive non-negative extension of K_0 in $\mathcal{H}$ other than K.

To express this criterium in terms of given K and A we use the following proposition.

Proposition.

Let L be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ *given as* 2×2 *block operator matrix,*

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where R and S are invertible operators in $\mathcal{M}$ *and* $\mathcal{N}$*, respectively, and* U, V *act between* $\mathcal{M}$ *and* $\mathcal{N}$*.*
If R is invertible operator in $\mathcal{M}$*, then*

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_{\mathcal{N}} \Lambda^{-1} P_{\mathcal{N}} L^{-1},$$

$$\Lambda = P_{\mathcal{N}} L^{-1}|_{\mathcal{N}}.$$

Proposition

Proposition.

Let L be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as 2×2 block operator matrix,

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where R and S are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and U, V act between $\mathcal{M}$ and $\mathcal{N}$.

If R is invertible operator in $\mathcal{M}$,then

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_{\mathcal{N}} \Lambda^{-1} P_{\mathcal{N}} L^{-1},$$

$$\Lambda = P_{\mathcal{N}} L^{-1}|_{\mathcal{N}}.$$

Proposition.

Let L be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as 2×2 block operator matrix,

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where R and S are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and U, V act between $\mathcal{M}$ and $\mathcal{N}$. If R is invertible operator in $\mathcal{M}$, then

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_{\mathcal{N}} \Lambda^{-1} P_{\mathcal{N}} L^{-1},$$

$$\Lambda = P_{\mathcal{N}} L^{-1}|_{\mathcal{N}}.$$

Proposition

Proposition.

Let L be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ *given as* 2×2 *block operator matrix,*

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where R and S are invertible operators in $\mathcal{M}$ *and* $\mathcal{N}$*, respectively, and* U, V *act between* $\mathcal{M}$ *and* $\mathcal{N}$*.*
If R is invertible operator in $\mathcal{M}$*,then*

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_{\mathcal{N}} \Lambda^{-1} P_{\mathcal{N}} L^{-1},$$

$$\Lambda = P_{\mathcal{N}} L^{-1}|_{\mathcal{N}}.$$

Set

$$\Lambda_{1,\varepsilon} = P_{\mathcal{N}}(K + \varepsilon I)^{-1}|_{\mathcal{N}}$$
$$\Lambda_{2,\varepsilon} = P_{\mathcal{N}}[(1+\varepsilon)I - K]^{-1}|_{\mathcal{N}}\,.$$

Applying the above Proposition with $L = K + \varepsilon I$ and

$$R = T + \varepsilon I\,,$$
$$U = \Gamma^* = P_{\mathcal{M}}K|_{\mathcal{N}} = P_{\mathcal{M}}[K + \varepsilon I]|_{\mathcal{N}}\,,$$
$$V = \Gamma = P_{\mathcal{N}}K|_{\mathcal{M}} = P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{M}}\,,$$
$$S = P_{\mathcal{N}}K|_{\mathcal{N}} + \varepsilon I$$

yields

$$\Gamma(T + \varepsilon I)^{-1}\Gamma^* = P_{\mathcal{N}}K|_{\mathcal{N}} + \varepsilon I - \Lambda_{1,\varepsilon}^{-1}.$$

Set

$$\Lambda_{1,\varepsilon} = P_{\mathcal{N}}(K+\varepsilon I)^{-1}|_{\mathcal{N}}$$
$$\Lambda_{2,\varepsilon} = P_{\mathcal{N}}[(1+\varepsilon)I - K]^{-1}|_{\mathcal{N}}\,.$$

Applying the above Proposition with $L = K + \varepsilon I$ and

$$R = T + \varepsilon I\,,$$
$$U = \Gamma^* = P_{\mathcal{M}}K|_{\mathcal{N}} = P_{\mathcal{M}}[K+\varepsilon I]|_{\mathcal{N}}\,,$$
$$V = \Gamma = P_{\mathcal{N}}K|_{\mathcal{M}} = P_{\mathcal{N}}[K+\varepsilon I]|_{\mathcal{M}}\,,$$
$$S = P_{\mathcal{N}}K|_{\mathcal{N}} + \varepsilon I$$

yields

$$\Gamma(T+\varepsilon I)^{-1}\Gamma^* = P_{\mathcal{N}}K|_{\mathcal{N}} + \varepsilon I - \Lambda_{1,\varepsilon}^{-1}.$$

Set

$$\Lambda_{1,\varepsilon} = P_{\mathcal{N}}(K+\varepsilon I)^{-1}|_{\mathcal{N}}$$
$$\Lambda_{2,\varepsilon} = P_{\mathcal{N}}[(1+\varepsilon)I - K]^{-1}|_{\mathcal{N}}\,.$$

Applying the above Proposition with $L = K + \varepsilon I$ and

$$R = T + \varepsilon I\,,$$
$$U = \Gamma^* = P_{\mathcal{M}} K|_{\mathcal{N}} = P_{\mathcal{M}}[K+\varepsilon I]|_{\mathcal{N}}\,,$$
$$V = \Gamma = P_{\mathcal{N}} K|_{\mathcal{M}} = P_{\mathcal{N}}[K+\varepsilon I]|_{\mathcal{M}}\,,$$
$$S = P_{\mathcal{N}} K|_{\mathcal{N}} + \varepsilon I$$

yields

$$\Gamma(T+\varepsilon I)^{-1}\Gamma^* = P_{\mathcal{N}} K|_{\mathcal{N}} + \varepsilon I - \Lambda_{1,\varepsilon}^{-1}.$$

In the same fashion we get

$$\Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* = P_{\mathcal{N}}[I-K]|_{\mathcal{N}} + \varepsilon I - \Lambda_{2,\varepsilon}^{-1}.$$

Hence

$$I - \lim_{\varepsilon\downarrow 0}\left(\Gamma(T+\varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^*\right) = \lim_{\varepsilon\downarrow 0}\Lambda_{1,\varepsilon}^{-1} + \lim_{\varepsilon\downarrow 0}\Lambda_{2,\varepsilon}^{-1}.$$

In the same fashion we get

$$\Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* = P_{\mathcal{N}}[I-K]|_{\mathcal{N}} + \varepsilon I - \Lambda_{2,\varepsilon}^{-1}.$$

Hence

$$I - \lim_{\varepsilon\downarrow 0}\left(\Gamma(T+\varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^*\right) = \lim_{\varepsilon\downarrow 0}\Lambda_{1,\varepsilon}^{-1} + \lim_{\varepsilon\downarrow 0}\Lambda_{2,\varepsilon}^{-1}.$$

Theorem.

Let K be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, K_0 is the restriction of K onto the subspace $\mathcal{M}(= \mathcal{M} \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{N}}\right)^{-1}$$

$$G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I - K + \varepsilon I]|_{\mathcal{N}}\right)^{-1}$$

Then the set $\mathbf{C}$ of all non-negative contractive extensions K_X of K_0 in $\mathcal{H}$ is described by expression

$$K_X = \begin{pmatrix} P_{\mathcal{M}}K|_{\mathcal{M}} & P_{\mathcal{M}}K|_{\mathcal{N}} \\ P_{\mathcal{M}}K|_{\mathcal{N}} & X \end{pmatrix}, \quad (1)$$

where X runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$P_{\mathcal{N}}K|_{\mathcal{N}} - G_1 \leq X \leq P_{\mathcal{N}}K|_{\mathcal{N}} + G_2. \quad (2)$$

In particular, K is the unique non-negative contractive extension of K_0 if and only if $G_1 = G_2 = 0$.

Theorem.

Let K be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, K_0 is the restriction of K onto the subspace $\mathcal{M} (= \mathcal{M} \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{N}}\right)^{-1}$$

$$G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I - K + \varepsilon I]|_{\mathcal{N}}\right)^{-1}$$

Then the set **C** *of all non-negative contractive extensions K_X of K_0 in $\mathcal{H}$ is described by expression*

$$K_X = \begin{pmatrix} P_{\mathcal{M}} K|_{\mathcal{M}} & P_{\mathcal{M}} K|_{\mathcal{N}} \\ P_{\mathcal{M}} K|_{\mathcal{N}} & X \end{pmatrix}, \tag{1}$$

where X runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$P_{\mathcal{N}} K|_{\mathcal{N}} - G_1 \leq X \leq P_{\mathcal{N}} K|_{\mathcal{N}} + G_2. \tag{2}$$

In particular, K is the unique non-negative contractive extension of K_0 if and only if $G_1 = G_2 = 0$.

Theorem.

Let K be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, K_0 is the restriction of K onto the subspace $\mathcal{M}(= \mathcal{M} \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{N}} \right)^{-1}$$

$$G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I - K + \varepsilon I]|_{\mathcal{N}} \right)^{-1}$$

*Then the set **C** of all non-negative contractive extensions K_X of K_0 in $\mathcal{H}$ is described by expression*

$$K_X = \begin{pmatrix} P_{\mathcal{M}}K|_{\mathcal{M}} & P_{\mathcal{M}}K|_{\mathcal{N}} \\ P_{\mathcal{M}}K|_{\mathcal{N}} & X \end{pmatrix}, \qquad (1)$$

where X runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$P_{\mathcal{N}}K|_{\mathcal{N}} - G_1 \leq X \leq P_{\mathcal{N}}K|_{\mathcal{N}} + G_2. \qquad (2)$$

In particular, K is the unique non-negative contractive extension of K_0 if and only if $G_1 = G_2 = 0$.

Theorem.

Let K be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, K_0 is the restriction of K onto the subspace $\mathcal{M}(= \mathcal{M} \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{N}}\right)^{-1}$$

$$G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I - K + \varepsilon I]|_{\mathcal{N}}\right)^{-1}$$

Then the set **C** *of all non-negative contractive extensions K_X of K_0 in $\mathcal{H}$ is described by expression*

$$K_X = \begin{pmatrix} P_{\mathcal{M}}K|_{\mathcal{M}} & P_{\mathcal{M}}K|_{\mathcal{N}} \\ P_{\mathcal{M}}K|_{\mathcal{N}} & X \end{pmatrix}, \tag{1}$$

where X runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$P_{\mathcal{N}}K|_{\mathcal{N}} - G_1 \leq X \leq P_{\mathcal{N}}K|_{\mathcal{N}} + G_2. \tag{2}$$

In particular, K is the unique non-negative contractive extension of K_0 if and only if $G_1 = G_2 = 0$.

Remark.

The set of all non-negative singular perturbations of A contains the minimal perturbation A_μ with and the maximal perturbation A_M such that any non-negative perturbation A_1 satisfies inequalities $A_\mu \leq A_1 \leq A_M$. The corresponding values of parameters X in the above theorem are

$$X_\mu = I|_{\mathcal{N}} + P_{\mathcal{N}}[I + A]^{-1}|_{\mathcal{N}} - G_1$$
$$X_M = I|_{\mathcal{N}} + P_{\mathcal{N}}[I + A]^{-1}|_{\mathcal{N}} + G_2$$

If $G_1 = 0\,(G_2 = 0)$, then the minimal (maximal) perturbation coincides with A.

Proposition.

The set of resolvents of all non-negative singular perturbations A_Y of A is described by the M.G. Krein formula

$$(A_Y - zI)^{-1} = (A - zI)^{-1} - (A + I)(A - zI)^{-1} P_{\mathcal{N}} Y \times$$

$$\left[I + (1 + z) P_{\mathcal{N}} (A + I)(A - zI)^{-1} Y \right]^{-1} \times P_{\mathcal{N}} (A + I)(A - zI)^{-1},$$

where Y runs contractions in $\mathcal{N}$ satisfying inequalities $-G_1 \leq Y \leq G_2$.

Let A denote the multiplication operator in $L_2(\mathbf{R}_n)$ by the continuous function $\varphi(k)$, $k^2 = k_1^2 + ... + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

$$\int_0^\infty \frac{k^{n-1}}{(1+\varphi(k))^2}\, dk < \infty.$$

A is a non-negative self-adjoint operator,

$$\mathcal{D}(A) = \left\{ f : \int_{\mathbf{R}_n} |1+\varphi(k)|^2 |f(\mathbf{k}|^2 d\mathbf{k} < \infty, \ f \in L_2(\mathbf{R}_n) \right\}.$$

Applications

Let A denote the multiplication operator in $L_2(\mathbf{R}_n)$ by the continuous function $\varphi(k)$, $k^2 = k_1^2 + ... + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

$$\int_0^\infty \frac{k^{n-1}}{(1+\varphi(k))^2}\, dk < \infty.$$

A is a non-negative self-adjoint operator,

$$\mathcal{D}(A) = \left\{ f : \int_{\mathbf{R}_n} |1+\varphi(k)|^2 |f(\mathbf{k}|^2 d\mathbf{k} < \infty,\ f \in L_2(\mathbf{R}_n) \right\}.$$

Applications

Let $\hat{\delta}$ denote the unbounded linear functional in $L_2(\mathbf{R}_n)$:

$$\hat{\delta}(f) = \int_{\mathbf{R}_n} f(\mathbf{k}) d\mathbf{k}.$$

Note that $\mathcal{D}(\hat{\delta}) \subset \mathcal{D}(A)$.

Let us denote by A_0 the restriction of A onto linear set

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0 \right\}.$$

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (L_2(\mathbf{R}_n) \ominus (I + A)\mathcal{D}_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)},\ \xi \in \mathbf{C} \right\}.$$

Applications

Let $\hat{\delta}$ denote the unbounded linear functional in $L_2(\mathbf{R}_n)$:

$$\hat{\delta}(f) = \int_{\mathbf{R}_n} f(\mathbf{k})d\mathbf{k}.$$

Note that $\mathcal{D}(\hat{\delta}) \subset \mathcal{D}(A)$.

Let us denote by A_0 the restriction of A onto linear set

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0 \right\}.$$

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (L_2(\mathbf{R}_n) \ominus (I + A)\mathcal{D}_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)},\ \xi \in \mathbf{C} \right\}.$$

Applications

Let $\hat{\delta}$ denote the unbounded linear functional in $L_2(\mathbf{R}_n)$:

$$\hat{\delta}(f) = \int_{\mathbf{R}_n} f(\mathbf{k}) d\mathbf{k}.$$

Note that $\mathcal{D}(\hat{\delta}) \subset \mathcal{D}(A)$.

Let us denote by A_0 the restriction of A onto linear set

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0 \right\}.$$

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (L_2(\mathbf{R}_n) \ominus (I + A)\mathcal{D}_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)},\ \xi \in \mathbf{C} \right\}.$$

Applications

Let $\hat{\delta}$ denote the unbounded linear functional in $L_2(\mathbf{R}_n)$:

$$\hat{\delta}(f) = \int_{\mathbf{R}_n} f(\mathbf{k}) d\mathbf{k}.$$

Note that $\mathcal{D}(\hat{\delta}) \subset \mathcal{D}(A)$.

Let us denote by A_0 the restriction of A onto linear set

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0 \right\}.$$

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (L_2(\mathbf{R}_n) \ominus (I + A)\mathcal{D}_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)},\ \xi \in \mathbf{C} \right\}.$$

Proposition.

A is the unique non-negative self-adjoint extension of A_0 that is A has no non-negative singular perturbations if and only if

$$\int_0^\infty \frac{k^{n-1}}{\varphi(k)(1+\varphi(k))}\, dk = \infty$$

and

$$\int_0^\infty \frac{k^{n-1}}{(1+\varphi(k))}\, dk = \infty.$$

Proposition.

A is the unique non-negative self-adjoint extension of A_0 that is A has no non-negative singular perturbations if and only if

$$\int_0^\infty \frac{k^{n-1}}{\varphi(k)(1+\varphi(k))}\, dk = \infty$$

and

$$\int_0^\infty \frac{k^{n-1}}{(1+\varphi(k))}\, dk = \infty.$$

Proposition.

A is the unique non-negative self-adjoint extension of A_0 that is A has no non-negative singular perturbations if and only if

$$\int_0^\infty \frac{k^{n-1}}{\varphi(k)(1+\varphi(k))}\,dk = \infty$$

and

$$\int_0^\infty \frac{k^{n-1}}{(1+\varphi(k))}\,dk = \infty.$$

Proposition.

A is the unique non-negative self-adjoint extension of A_0 *that is A has no non-negative singular perturbations if and only if*

$$\int_0^\infty \frac{k^{n-1}}{\varphi(k)(1+\varphi(k))}\,dk = \infty$$

and

$$\int_0^\infty \frac{k^{n-1}}{(1+\varphi(k))}\,dk = \infty.$$

Put $\varphi(k) = k^2$ and let $n = 2$.

Corollary.

The self-adjoint Laplace operator in $L_2(\mathbf{R}_2)$ has no non-negative singular perturbations with support at one point of $\mathbf{R}_2$.

The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbf{R}_2)$ with support at two or more points do exist. Let us consider the restriction A_0 of the multiplication operator operator by k^2, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

$$e_0(\mathbf{k}) = \frac{1 - \exp(-i(\mathbf{k}\cdot\mathbf{x}_0))}{1+k^2}, \ \mathbf{x}_0 \in \mathbf{R}_2.$$

In this case

$$\|e_0\|^2 = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{(1+k^2)^2}\, d\mathbf{k} < \infty,$$

$$((I+A)A^{-1}e_0, e_0) = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{k^2(1+k^2)}\, d\mathbf{k} < \infty,$$

$$((I+A)e_0, e_0) = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{1+k^2}\, d\mathbf{k} = \infty.$$

Hence $G_1 = \|e_0\|^2 \cdot ((I+A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$.

The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbf{R}_2)$ with support at two or more points do exist. Let us consider the restriction A_0 of the multiplication operator operator by k^2, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

$$e_0(\mathbf{k}) = \frac{1 - \exp(-i(\mathbf{k} \cdot \mathbf{x}_0))}{1 + k^2}, \ \mathbf{x}_0 \in \mathbf{R}_2.$$

In this case

$$\|e_0\|^2 = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{(1 + k^2)^2} \, d\mathbf{k} < \infty,$$

$$((I + A)A^{-1} e_0, e_0) = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{k^2(1 + k^2)} \, d\mathbf{k} < \infty,$$

$$((I + A)e_0, e_0) = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{1 + k^2} \, d\mathbf{k} = \infty\,.$$

Hence $G_1 = \|e_0\|^2 \cdot ((I + A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$.

The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbf{R}_2)$ with support at two or more points do exist. Let us consider the restriction A_0 of the multiplication operator operator by k^2, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

$$e_0(\mathbf{k}) = \frac{1 - \exp(-i(\mathbf{k} \cdot \mathbf{x}_0))}{1 + k^2}, \; \mathbf{x}_0 \in \mathbf{R}_2.$$

In this case

$$\|e_0\|^2 = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{(1 + k^2)^2} \, d\mathbf{k} < \infty,$$

$$((I + A)A^{-1} e_0, e_0) = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{k^2(1 + k^2)} \, d\mathbf{k} < \infty,$$

$$((I + A)e_0, e_0) = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{1 + k^2} \, d\mathbf{k} = \infty.$$

Hence $G_1 = \|e_0\|^2 \cdot ((I + A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$.

The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbf{R}_2)$ with support at two or more points do exist. Let us consider the restriction A_0 of the multiplication operator operator by k^2, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

$$e_0(\mathbf{k}) = \frac{1 - \exp(-i(\mathbf{k}\cdot\mathbf{x}_0))}{1+k^2}, \ \mathbf{x}_0 \in \mathbf{R}_2.$$

In this case

$$\|e_0\|^2 = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{(1+k^2)^2}\, d\mathbf{k} < \infty,$$

$$((I+A)A^{-1}e_0, e_0) = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{k^2(1+k^2)}\, d\mathbf{k} < \infty,$$

$$((I+A)e_0, e_0) = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{1+k^2}\, d\mathbf{k} = \infty\,.$$

Hence $G_1 = \|e_0\|^2 \cdot ((I+A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$.

The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbf{R}_2)$ with support at two or more points do exist. Let us consider the restriction A_0 of the multiplication operator operator by k^2, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

$$e_0(\mathbf{k}) = \frac{1 - \exp(-i(\mathbf{k}\cdot\mathbf{x}_0))}{1+k^2}, \ \mathbf{x}_0 \in \mathbf{R}_2.$$

In this case

$$\|e_0\|^2 = \int_{\mathbf{R}_2} \frac{4\sin^2 \frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{(1+k^2)^2}\, d\mathbf{k} < \infty,$$

$$((I+A)A^{-1}e_0, e_0) = \int_{\mathbf{R}_2} \frac{4\sin^2 \frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{k^2(1+k^2)}\, d\mathbf{k} < \infty,$$

$$((I+A)e_0, e_0) = \int_{\mathbf{R}_2} \frac{4\sin^2 \frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{1+k^2}\, d\mathbf{k} = \infty\,.$$

Hence $G_1 = \|e_0\|^2 \cdot ((I+A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$.

As follows, the concerned restriction A_0 of the multiplication operator A by k^2 has non-negative self-adjoint extensions in $L_2\,(\,\mathbf{R}_2\,)$ others then A and A is the maximal element in the set of these extensions.

It remains to note that A is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $L_2(\mathbf{R}_2)$ and A_0 is isomorphic to the restriction of this $-\Delta$ on the subset of function $f(\mathbf{x})$ from $\mathcal{D}(-\Delta)$ satisfying conditions:

$$\lim_{|\mathbf{x}|\to 0}(\ln|\mathbf{x}|)^{-1}f(\mathbf{x}) - \lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}-\mathbf{x}_0|)^{-1}f(\mathbf{x}) = 0,$$

$$\lim_{|\mathbf{x}|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}|\lim_{|\mathbf{x}'|\to 0}(\ln|\mathbf{x}'|)^{-1}f(\mathbf{x}')\right] -$$
$$\lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}-\mathbf{x}_0|\lim_{|\mathbf{x}'-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}'-\mathbf{x}_0|)^{-1}f(\mathbf{x}')\right] = 0.$$

It remains to note that A is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $L_2(\mathbf{R}_2)$ and A_0 is isomorphic to the restriction of this $-\Delta$ on the subset of function $f(\mathbf{x})$ from $\mathcal{D}(-\Delta)$ satisfying conditions:

$$\lim_{|\mathbf{x}|\to 0}(\ln|\mathbf{x}|)^{-1}f(\mathbf{x}) - \lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}-\mathbf{x}_0|)^{-1}f(\mathbf{x}) = 0,$$

$$\lim_{|\mathbf{x}|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}|\lim_{|\mathbf{x}'|\to 0}(\ln|\mathbf{x}'|)^{-1}f(\mathbf{x}')\right] -$$

$$\lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}-\mathbf{x}_0|\lim_{|\mathbf{x}'-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}'-\mathbf{x}_0|)^{-1}f(\mathbf{x}')\right] = 0.$$

It remains to note that A is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $L_2(\mathbf{R}_2)$ and A_0 is isomorphic to the restriction of this $-\Delta$ on the subset of function $f(\mathbf{x})$ from $\mathcal{D}(-\Delta)$ satisfying conditions:

$$\lim_{|\mathbf{x}|\to 0}(\ln|\mathbf{x}|)^{-1}f(\mathbf{x}) - \lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}-\mathbf{x}_0|)^{-1}f(\mathbf{x}) = 0,$$

$$\lim_{|\mathbf{x}|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}|\lim_{|\mathbf{x}'|\to 0}(\ln|\mathbf{x}'|)^{-1}f(\mathbf{x}')\right] -$$
$$\lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}-\mathbf{x}_0|\lim_{|\mathbf{x}'-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}'-\mathbf{x}_0|)^{-1}f(\mathbf{x}')\right] = 0.$$

The self-adjoint Laplace operator in $L_2\,(\,\mathbf{R}_3\,)$ has infinitely many non-negative singular perturbations with support at one point of $\mathbf{R}_3$ and the standardly defined Laplace the maximal element in the set of this perturbation.

Consider the multiplication operator A by k^{2l} in $L_2(\mathbf{R}_n)$ assuming that $4l \leq n+1$. A is isomorphic to the polyharmonic operator $(-\Delta)^l$ in $L_2(\mathbf{R}_n)$.

Let us consider the restriction A_0 of A with the domain

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0 \right\}.$$

that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator $(-\Delta)^l$ onto the Sobolev subspace $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Consider the multiplication operator A by k^{2l} in $L_2\left(\mathbf{R}_n\right)$ assuming that $4l \leq n+1$. A is isomorphic to the polyharmonic operator $(-\Delta)^l$ in $L_2\left(\mathbf{R}_n\right)$.

Let us consider the restriction A_0 of A with the domain

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0 \right\}.$$

that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator $(-\Delta)^l$ onto the Sobolev subspace $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Propositions

Proposition.

If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction A_0 and $(-\Delta)^l$ is the minimal element in the set of the non-negative extensions of A_0 in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n > 2l$ then there is the infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with A_0 and for those as non-negative extensions of A_0 in the set of the in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$ the operator $(-\Delta)^l$ is the maximal element.

Propositions

Proposition.

If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction A_0 and $(-\Delta)^l$ is the minimal element in the set of the non-negative extensions of A_0 in $H_{2l}^2(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $H_{2l}^2(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n > 2l$ then there is the infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with A_0 and for those as non-negative extensions of A_0 in the set of the in $H_{2l}^2(\mathbf{R}_n \setminus \{0\})$ the operator $(-\Delta)^l$ is the maximal element.

Propositions

Proposition.

If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction A_0 and $(-\Delta)^l$ is the minimal element in the set of the non-negative extensions of A_0 in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n > 2l$ then there is the infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with A_0 and for those as non-negative extensions of A_0 in the set of the in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$ the operator $(-\Delta)^l$ is the maximal element.

Propositions

Proposition.

If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction A_0 and $(-\Delta)^l$ is the minimal element in the set of the non-negative extensions of A_0 in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n > 2l$ then there is the infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with A_0 and for those as non-negative extensions of A_0 in the set of the in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$ the operator $(-\Delta)^l$ is the maximal element.

Propositions

Proposition.

If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction A_0 and $(-\Delta)^l$ is the minimal element in the set of the non-negative extensions of A_0 in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

Proposition.

If $n > 2l$ then there is the infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with A_0 and for those as non-negative extensions of A_0 in the set of the in $H^2_{2l}(\mathbf{R}_n \setminus \{0\})$ the operator $(-\Delta)^l$ is the maximal element.