# CARATHÉODORY FUNCTIONS IN THE BANACH SPACE SETTING

**Daniel Alpay***
**Department of mathematics**
**Ben–Gurion University of the Negev**
**Beer-Sheva**
**Israel**

**Joint work with Olga Timoshenko and Dan Volok**

*Earl Katz Family Chair in algebraic system theory

# Outline of the talk

1. The Hilbert space case

2. Review of positive operators in Banach spaces

3. The main results (representation theorems for Carathéodory functions in the setting of Banach space).

4. Conclusions

## The classical case

Various families of reproducing kernel Hilbert spaces of functions which take values in a Hilbert space and are analytic in the open unit disk or in the open upper half-plane have been introduced by L. de Branges and J. Rovnyak.

These spaces play an important role in operator theory, interpolation theory, inverse scattering, the theory of wide sense stationary stochastic processes and related topics

**In the case of the open unit disk $\mathbb{D}$, of particular importance are:**

$$k_\phi(z,w) = \frac{\phi(z)+\phi(w)^*}{2(1-zw^*)},$$
$$k_s(z,w) = \frac{I-s(z)s(w)^*}{1-zw^*}.$$

$s(z)$ **and** $\phi(z)$ **are operator-valued functions analytic in** $\mathbb{D}$ **are called Carathéodory and Schur functions when the kernels are positive.**

**The Cayley transform**

$$s(z) = (I-\phi(z))(I+\phi(z))^{-1}$$

**reduces the study of the kernels** $k_\phi(z,w)$ **to the study of the kernels** $k_s(z,w)$**. For these latter it is well known that the positivity of the kernel** $k_s(z,w)$ **implies analyticity of** $s(z)$**.**

**Every Carathéodory function admits two equivalent representations. The first, (Riesz – Herglotz representation) is:**

$$\phi(z) = ia + \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)$$

**where $a$ is a self-adjoint operator and where $\mu(t)$ is an increasing function such that**

**$\mu(2\pi) < \infty$.**

**The integral is a Stieltjes integral and the proof relies on Helly's theorem.**

**The second representation is:**

$$\phi(z) = ia + \Gamma(U + zI)(U - zI)^{-1}\Gamma^*$$

**where $U$ is a unitary operator in an auxiliary Hilbert space $\mathcal{H}$ and $\Gamma$ is a bounded operator from $\mathcal{H}$ into $\mathcal{C}$ (the coefficient space).**

The expression

$$\boxed{\phi(z) = ia + \Gamma(U + zI)(U - zI)^{-1}\Gamma^*}$$

still makes sense in a more general setting when the kernel $k_\phi(z, w)$ has a finite number of negative squares. The space $\mathcal{H}$ is then a Pontryagin space. (see Kreĭn and Langer).

They allowed the values of the function $\phi(z)$ to be operators between Pontryagin spaces and required weak continuity at the origin.

**Without some continuity hypothesis one can find functions for which the kernel**

$k_\phi(z,w) = \frac{\phi(z)+\phi(w)^*}{2(1-zw^*)}$

**has a finite number of negative squares but which are not meromorphic in $\mathbb{D}$ and cannot admit representations of the form above. For instance the function**

$$\phi(z) = \begin{cases} 0 \text{ if } z \neq 0 \\ 1 \text{ if } z = 0 \end{cases}$$

**defines a kernel $k_\phi(z,w)$ which has one negative square.**

The notion of reproducing kernel space (with positive or indefinite metric) can also be introduced for functions which take values in Banach spaces. Motivations originate from the theory of partial differential equations and the theory of stochastic processes.

| |
|---|
| It seems that there are no natural analogs of Schur functions or the Cayley transform in this setting. |

## The Banach space setting: Positive operators and positive kernels

Let $\mathcal{B}$ be a Banach space.

$\mathcal{B}^*$ is the space of anti-linear bounded functionals (that is, its conjugate dual space).

Duality between $\mathcal{B}$ and $\mathcal{B}^*$:

$$\langle b_*, b\rangle_{\mathcal{B}} = b_*(b), \quad \text{where } b \in \mathcal{B} \text{ and } b_* \in \mathcal{B}^*.$$

The operator $A \in \mathrm{L}(\mathcal{B}, \mathcal{B}^*)$ is positive if

$$\langle Ab, b\rangle_{\mathcal{B}} \geq 0, \quad \forall b \in \mathcal{B}.$$

Note that a positive operator is in particular self-adjoint in the sense that $A = A^* \Big|_{\mathcal{B}}$, that is,

$$\langle Ab, c\rangle_{\mathcal{B}} = \overline{\langle Ac, b\rangle_{\mathcal{B}}}.$$

**The following factorization result is well known and originates with the works of Pedrick (in the case of topological vector spaces with appropriate properties) and Vakhania (for positive elements in $\mathrm{L}(\mathcal{B}^*, \mathcal{B})$)**

**Theorem:** *The operator $A \in \mathcal{L}(\mathcal{B}, \mathcal{B}^*)$ is positive if and only if there exist a Hilbert space $\mathcal{H}$ and a bounded operator $T \in \mathrm{L}(\mathcal{B}, \mathcal{H})$ such that $A = T^*T$. Moreover,*

$$\langle Ab, c\rangle_{\mathcal{B}} = \langle Tb, Tc\rangle_{\mathcal{H}}, \quad b, c \in \mathcal{B}$$

*and*

$$\sup_{\|b\|=1} \langle Ab, b\rangle_{\mathcal{B}} = \|A\| = \|T\|^2.$$

*Finally*

$$|\langle Ab, c\rangle_{\mathcal{B}}| \leq \langle Ab, b\rangle^{1/2} \langle Ac, c\rangle^{1/2}.$$

**We will say that $A \leq B$ if $B - A \geq 0$. Note that**

$$A \leq B \implies \|A\| \leq \|B\|.$$

Let $\mathcal{H}$ be a Hilbert space of $\mathcal{B}^*$-valued functions defined on a set $\Omega$ and let $K(z,w)$ be an $\mathrm{L}(\mathcal{B},\mathcal{B}^*)$-valued kernel defined on $\Omega \times \Omega$. The kernel $K(z,w)$ is called the reproducing kernel of the Hilbert space $\mathcal{H}$ if for every $w \in \Omega$ and $b \in \mathcal{B}$ $K(\cdot,w)b \in \mathcal{H}$ and

$$\langle f, K(\cdot,w)b\rangle_{\mathcal{H}} = \langle f(w), b\rangle_{\mathcal{B}}, \qquad \forall f \in \mathcal{H}.$$

Let $K(z,w)$ be an $\mathrm{L}(\mathcal{B},\mathcal{B}^*)$-valued kernel defined on $\Omega \times \Omega$. The kernel $K(z,w)$ is said to be positive if for any choice of $z_1,\ldots,z_n \in \Omega$ and $b_1,\ldots,b_n \in \mathcal{B}$ it holds that

$$\sum_{j=1}^{n} \langle K(z_i,z_j)b_j, b_i\rangle_{\mathcal{B}} \geq 0.$$

The reproducing kernel $K(z,w)$ of a Hilbert space of $\mathcal{B}^*$-valued functions, when it exists, is unique and positive.

**Let $K(z,w)$ be an $\mathrm{L}(\mathcal{B},\mathcal{B}^*)$-valued positive kernel defined on $\Omega\times\Omega$. Then there exists a unique Hilbert space of $\mathcal{B}^*$-valued functions defined on $\Omega$ with the reproducing kernel $K(z,w)$.**

**One can derive the notion of a reproducing kernel Hilbert space of $\mathcal{B}$-valued functions using the natural injection $\tau$ from $\mathcal{B}$ into $\mathcal{B}^{**}$ defined by**

$$\langle \tau(b), b_* \rangle_{\mathcal{B}^*} = \overline{\langle b_*, b \rangle_{\mathcal{B}}}.$$

## Stieltjes integral

**Given an increasing positive function**

$$M\ :\ [a,b] \longrightarrow \mathbf{L}(\mathcal{B},\mathcal{B}^*).$$

**Thus,** $M(t) \geq 0$ **for all** $t \in [a,b]$ **and moreover**

$$a \leq t_1 \leq t_2 \leq b \Longrightarrow M(t_2) - M(t_1) \geq 0.$$

**Let** $f(t)$ **be a scalar continuous function on** $[a,b]$ **and let**

$$a = t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq t_2 \leq \cdots \leq \xi_m \leq t_m = b$$

**be a subdivision of** $[a,b]$ **be a decomposition of** $[a,b]$**.**

**The Stieltjes integral** $\int_a^b f(t)dM(t)$ **is the limit (in the** $\mathrm{L}(\mathcal{B},\mathcal{B}^*)$ **topology) of the sums of the form**

$$\sum_{j=1}^{m} f(\xi_j)(M(t_j) - M(t_{j-1}))$$

**as** $\sup_j |t_j - t_{j-1}|$ **goes to** 0**.**

**Theorem: The integral** $\int_a^b f(t)dM(t)$ **exists.**

## Helly's theorem

**Theorem: Let $F_n(t)$ ($t \in [0, 2\pi]$) be a sequence of increasing $\mathrm{L}(\mathcal{B}, \mathcal{B}^*)$–valued functions such that**

$$F_n(t) \leq F_0, \quad n = 0, 1, \dots \quad \text{and} \quad t \in [0, 2\pi],$$

**where $F_0$ is some bounded operator. Then, there exists a subsequence of $F_n$ which converges weakly for every $t \in [0, 2\pi]$. Moreover, for $f(t)$ a continuous scalar function we have (in the weak sense, and via the subsequence):**

$$\int_0^{2\pi} f(t) dF(t) = \lim_{n \to \infty} f(t) dF_n(t)$$

**($\mathcal{B}$ is separable)**

## $\mathrm{L}(\mathcal{B}, \mathcal{B}^*)$-valued Carathéodory functions

**An $\mathrm{L}(\mathcal{B}, \mathcal{B}^*)$-valued function $\phi(z)$ weakly continuous at the origin will be called a Carathéodory function if the $\mathrm{L}(\mathcal{B}, \mathcal{B}^*)$-valued kernel**

$$k_\phi(z,w) = \frac{\phi(z)+\phi(w)^*\Big|_{\mathcal{B}}}{2(1-zw^*)}$$

**is positive.**

**We always weak continuity at the origin in the sense that**

$$\langle \phi(z)b, b\rangle_{\mathcal{B}} \to \langle \phi(0)b, b\rangle_{\mathcal{B}} \text{ as } z \to 0, \qquad \forall b \in \mathcal{B}.$$

For a Carathéodory function $\phi(z)$ we shall denote by $\mathcal{L}(\phi)$ the Hilbert space of $\mathcal{B}^*$-valued functions with the reproducing kernel

$$k_\phi(z,w) = \frac{\phi(z)+\phi(w)^*\Big|_{\mathcal{B}}}{2(1-zw^*)}$$

We give two representation theorems for Carathéodory functions. In the first we make no assumption on the space $\mathcal{B}$. Following arguments of Krein and Langer, we prove the existence of a realization of the form

$$\phi(z)^*\Big|_{\mathcal{B}} = D + C^*(I - z^*V)^{-1}(I + z^*V)^{-1}C$$

The second theorem assumes that the space $\mathcal{B}$ is separable. We prove that in this case the Carathéodory functions can be characterized as functions analytic in the open unit disk with positive real part. Then we derive a Herglotz-type representation formula.

**Theorem:** Let $\phi(z)$ be an $\mathrm{L}(\mathcal{B},\mathcal{B}^*)$-valued defined in a neighborhood of the origin and weakly continuous at the origin. Then $\phi(z)$ is a Carathéodory function if and only if it admits the representation

$$\phi(z)^*\Big|_{\mathcal{B}} = D + C^*(I - z^*V)^{-1}(I + z^*V)^{-1}C$$

where $V$ is an isometric operator in some Hilbert space $\mathcal{H}$, $C$ is a bounded operator from $\mathcal{B}$ into $\mathcal{H}$ and $D$ is a purely imaginary operator from $\mathcal{B}$ into $\mathcal{B}^*$ in the sense that

$$D + D^*\Big|_{\mathcal{B}} = 0.$$

In particular, every Carathéodory function has an analytic extension to the whole open unit disk.

**Let $\phi(z)$ be a Carathéodory function. The elements of $\mathcal{L}(\phi)$ are weakly continuous at the origin:**

$$\langle f(w), b\rangle_{\mathcal{B}} \to \langle f(0), b\rangle_{\mathcal{B}} \text{ as } w \to 0,$$

$$\forall\, f \in \mathcal{L}(\phi), b \in \mathcal{B}.$$

**This is a consequence of the Cauchy − Schwarz inequality as**

$$\langle f(w), b\rangle_{\mathcal{B}} - \langle f(0), b\rangle_{\mathcal{B}} = \langle f, (k_\phi(\cdot, w) - k_\phi(\cdot, 0))b\rangle_{\mathcal{L}(\phi)}$$

**and**

$$\|(k_\phi(\cdot, w) - k_\phi(\cdot, 0))b\|^2_{\mathcal{L}(\phi)} = \frac{|w|^2}{1 - |w|^2}\Re\langle \phi(w)b, b\rangle_{\mathcal{B}}.$$

We consider in $\mathcal{L}(\phi) \times \mathcal{L}(\phi)$ the linear relation $R$ spanned by the pairs

$$\Big( \sum K_\phi(z, w_i) w_i^* b_i, \sum K_\phi(z, w_i) b_i - K_\phi(z, 0)(\sum b_i) \Big)$$

where all the sums are finite. This relation is densely defined because of the weak continuity of the elements of $\mathcal{L}(\phi)$ at the origin.

The relation $R$ is isometric. Its closure is thus the graph of an isometry, which we call $V$. We have:

$$V(K_\phi(z, w) w^* b) = K(z, w)b - K(z, 0)b,$$

and in particular

$$(I - w^* V)^{-1} K_\phi(\cdot, 0) b = K_\phi(\cdot, w) b.$$

Denote by $C$ the map

$$C : \mathcal{B} \longrightarrow \mathcal{L}(\phi), \quad (Cb)(z) = K(z, 0)b.$$

to obtain the required formula.

The arguments are very close to the ones in Krein-Langer, but we note the following:

we use a concrete space (the space $\mathcal{L}(\phi)$) to build the relation rather than abstract elements and the relation $R$ is defined slightly differently.

As already mentioned, the above argument still goes through when the kernel has a finite number of negative squares. In this case the space $\mathcal{L}(\phi)$ is a Pontryagin space.

**Let $\mathcal{B}$ be a separable Banach space and let $\phi(z)$ be a $\mathrm{L}(\mathcal{B}, \mathcal{B}^*)$–valued function analytic in the open unit disk, such that**

$$\phi(z) + \phi(z)^* \Big|_{\mathcal{B}} \geq 0.$$

**Then there exists an increasing $\mathrm{L}(\mathcal{B}, \mathcal{B}^*)$–valued function $M(t)$ ($t \in [0, 2\pi]$) and a purely imaginary operator $D$ such that**

$$\phi(z) = D + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dM(t),$$

**where the integral is defined in the weak sense. Furthermore the kernel $k_\phi(z, w)$ is positive in $\mathbb{D}$.**

**We follow the arguments in the book of Brodski, and apply Helly's theorem. The separability hypothesis of $\mathcal{B}$ is used at this point.**

## Conclusions:

**No Schur functions and no Cayley transform here.**

**Also results with unitary operators**

**Can define inverse scattering, interpolation, and related problems.**

**Look for applications to stochastic processes.**

**Also more general settings (Arveson space,...).**