Pontryagin theorem and an analysis of the spectral stability of solitons

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1. The main result.

2. Motivations.

3. Models.

Theorem 1. *Let* $\{\mathcal{H}, (\cdot,\cdot)\}$ *be a Hilbert space, let* $A = BG$, *where* $B, G : \mathcal{H} \to \mathcal{H}$ *are bounded operators,* $B:\ \operatorname{Im}(Bx, x) \geq 0$, $G = G^*$, $0 \notin \sigma_p(B) \cup \sigma_p(G)$, *and* $(-\infty, 0) \cap \sigma(G)$ *consists of a* $\kappa < \infty$ *eigenvalues.*
Then A *has a* κ*-dimensional* G*-nonpositive invariant subspace* $\mathcal{L}$ *such that* $\operatorname{Im}\sigma(A|\mathcal{L}) \geq 0$.

If $B = B^*$ *then there exists also an* A*-invariant* κ*-dimensional* G*-nonpositive subspace* $\mathcal{M}$ *such that* $\operatorname{Im}\sigma(A|\mathcal{M}) \leq 0$.

Proof. $G = J|G|$, $\{\widetilde{\mathcal{H}}, (\cdot,\cdot)_1\}$, $\mathcal{H} \subset \widetilde{\mathcal{H}}$, $(x, y)_1 = (|G|x, y)$, $x, y \in \mathcal{H}$

$$\widetilde{A} \supset A, \quad \widetilde{G} \supset G, \quad \widetilde{J} \supset J \implies \widetilde{A} = B\widetilde{G}.$$

$$\Pi_\kappa = \{\widetilde{\mathcal{H}}, [\cdot,\cdot] = (\widetilde{J}\cdot,\cdot)_1\}, \quad [x, y] = (Gx, y),\ x, y \in \mathcal{H}.$$

$\widetilde{A}$: $\Pi_\kappa \to \Pi_\kappa$ diss. $\Longrightarrow$ $\exists \mathcal{L}$, $\widetilde{A}\mathcal{L} \subset \mathcal{L}$, $\dim \mathcal{L} = \kappa$, $[x, x] \leq 0$, $x \in \mathcal{L}$ and $\operatorname{Im} \sigma(A|\mathcal{L}) \geq 0$.

$0 \notin \sigma_p(B) \cup \sigma_p(G) \Longrightarrow \widetilde{A}\mathcal{L} = \mathcal{L}$,

$\widetilde{A} = B\widetilde{G} \Longrightarrow \mathcal{L} \subset \mathcal{H}$ and $A\mathcal{L} = \mathcal{L}$.

$B = B^* \Longrightarrow \exists \mathcal{M}$, $\widetilde{A}\mathcal{M} = \mathcal{M}$ and $\operatorname{Im} \sigma(A|\mathcal{M}) \leq 0$.

Example 1. Let $\mathcal{H} = \operatorname{span}\{e_0\} \oplus H_1$, $\|e_0\| = 1$.

$$G = \begin{bmatrix} 0 & (\cdot, e)e_0 \\ (\cdot, e_0)e & G_1 \end{bmatrix}, \quad G_1 > 0,\ G_1 \in \mathfrak{S}_\infty,\ e \notin \operatorname{ran} G_1.$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

The operator G has $\kappa = 1$ negative eigenvalue, but

$$A = BG = \begin{bmatrix} 0 & 0 \\ (\cdot, e_0)e & G_1 \end{bmatrix}$$

has no G-nonpositive eigenvectors.

M.Grillakis, "Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system", Comm. Pure Appl. Math. **43**, 299–333 (1990) **[G]**

$$K(z) = R - zS$$

$R = R^*$, $S = S^*$, $0 \notin \sigma_p(R) \cup \sigma_p(S)$

$\exists H = H^* \gg 0 : R = H + W,\ S^{-1} = H + V$, W, V are H-compact.

$B := R^{-1}$, $G := S$, and $A := R^{-1}S$.

In **[G]** spectral properties and the stability problem for such a pencil $\equiv A = BG$ are studied.

$$\frac{du}{dt} = JE'(u(t)), \ u(t) \in \mathcal{H},$$

$E : \mathcal{H} \to \mathbb{R}$, $E \in C^2$, $J : \mathcal{H} \to \mathcal{H}$, $J = -J^{-1} = -J^*$.

Let $\varphi \in \mathcal{H}$ be a localized critical point (soliton): $E'(\varphi) = 0$.

Solitons exist for some nonlinear Schrödinger, Klein-Gordon, Korteweg-de Vries equations.

Linearization around φ:

$$\frac{dv}{dt} = JE''(\varphi)v + O(\|v\|^2).$$

Assume

$$\mathfrak{A} := JE'' = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \qquad L_\pm = L_\pm^*.$$

P1 $\sigma_c(L_\pm) \geq \omega_\pm$, $\omega_+ \geq 0$, $\omega_- > 0$;
P2 $\sigma_p(L_\pm)$ is a finite set (counting multiplicity)
P3 $\ker L_- \subset \operatorname{dom} L_+$ (**[G]**: $\operatorname{dom} L_+ = \operatorname{dom} L_-$).

Hence one can reduce the eigenvalue problem

$$\mathfrak{A}(u,w)^t = \lambda(u,w)^t \Longleftrightarrow L_+ u = -\lambda w,\ L_- w = \lambda u$$

to the case $\ker L_- = \{0\}$:

$$Ru = zSu, \quad R := L_+,\ S := L_-^{-1},\ z = -\lambda^2$$

We shown $\exists$ a regular point (z_0) of the pencil $K(z) = R - zS$.

$z = \frac{1}{\gamma} + z_0$: $Ru = zSu \Longrightarrow (R - z_0 S)^{-1} Su = \gamma u$

$B := (R - z_0 S)^{-1}$, $G := S$ and $A = (R - z_0 S)^{-1} S$.

Let $z_0 = 0$.

By definition, an eigenvalue z of the pencil $K(z)$ is **stable** iff $z > 0$ and semi-simple. Otherwise it is **unstable**.

Proposition 2. $K(z)$ has (i) a finite number (≥ 0) of negative eigenvalues and (ii) not more than $2\kappa + 1$ (counting multiplicity) nonreal and unstable positive eigenvalues.

(i) follows from assumptions **P1** and **P2**

(ii) from Theorem 1: $\sigma_{nonreal}(A) = \sigma_{nonreal}(A|\mathcal{L}) \cup \sigma_{nonreal}(A|\mathcal{L})^*$.

Proposition 3. If R and S have different (finite) numbers (counting multiplicity) of negative eigenvalues, the pencil $K(z)$ has at least one negative eigenvalue, that is, at least one unstable eigenvalue.

Example 2. Consider a scalar nonlinear Schrödinger equation:

$$i\psi_t = -\Delta\psi + F(|\psi|^2)\psi, \qquad \Delta = \partial^2_{x_1 x_1} + ... + \partial^2_{x_d x_d},$$

$(x,t) \in \mathbb{R}^d \times \mathbb{R}$, $\psi \in \mathbb{C}$.

If $F(|\psi|^2)$ is a localized potential, for instance, $F \in C^\infty$ and $F(0) = 0$, the equation has a solitary wave solution

$$\psi = \varphi(x) e^{i\omega t}, \quad \omega > 0, \ \varphi : \mathbb{R}^d \to \mathbb{R}$$

Hence $\varphi(x) \in C^\infty$ is an exponentially decreasing function.

(see, K. McLeod, "Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$", Trans. Amer. Math.Soc. **339**, 495–505 (1993)).

Substitute

$$\psi = \left(\varphi(x) + [u(x) + iw(x)]e^{\lambda t} + [\bar{u}(x) + i\bar{w}(x)]e^{\bar{\lambda} t}\right) e^{i\omega t},$$

$\lambda \in \mathbb{C}$ and $(u, w) \in \mathbb{C}^2$, and have a Hamiltonian system with Schrödinger operators:

$$L_+ = -\Delta + \omega + F(\varphi^2) + 2\varphi^2 F'(\varphi^2),$$

$$L_- = -\Delta + \omega + F(\varphi^2).$$

Here $L_\pm$ are unbounded operators,

$$\sigma_c(L_\pm) = [\omega, \infty), \quad \omega_+ = \omega_- = \omega > 0.$$

$$\dim \ker L_- \geq 1, \quad \varphi(x) \in \ker L_-,$$

$$\dim \ker L_+ \geq d, \quad \partial_{x_j}\varphi(x) \in \ker L_+, j = 1, ..., d.$$

$$\mathcal{H} := W_2^1(\mathbb{R}^d, \mathbb{C}).$$

Assumptions **P1** and **P2** hold since the functions $F(\varphi^2)$ and $\varphi^2 F'(\varphi^2)$ are exponentially decreasing.