On the eigenvalues of non-canonical self-adjoint extensions

Jussi Behrndt

jointly with Annemarie Luger

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$*$ Eigenvalues of $\widetilde{A}$ $=$ generalized zeros of $\lambda \mapsto m(\lambda) + \tau$.

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$*$ $\mu = \infty$ [LangerTextorius77, DerkachHassiMalamudSnoo00]

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$$f := P_{L^2}\big(\widetilde{A} - \lambda\big)^{-1} \restriction_{L^2} g.$$

* $*$ Spectral properties of $\widetilde{A}$ correspond to solvability of BVP, e.g. Nontrivial solutions of homogeneous problem $=$ Eigenvectors of $\widetilde{A}$.
* $*$ Good news: $\sigma_p(\widetilde{A})$ can be described with m and τ.

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We will only consider the case of a matrix-function τ !

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Proposition τ has generalized value at $\mu \iff \exists$ interval Δ, $\mu \in \Delta$:

$$\tau(\lambda) = \int_\Delta \frac{1}{t - \lambda} d\Sigma(t) + H_\Delta(\lambda), \quad \int_\Delta \frac{1}{(t-\mu)^2} d\Sigma(t) < \infty,$$

H_Δ holomorphic on Δ, Σ nondecreasing, left-cont. $n \times n$-matrix fct.

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