Diagonalization of the Coupled-Mode System.

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Coupled mode system with symmetries.

General symmetric 1-D coupled-mode system:

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, \bar{u}, v, \bar{v}) \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, \bar{u}, v, \bar{v}) \end{cases}$$

- **P1:** W is invariant with respect to the gauge transformation: $(u, v) \mapsto e^{i\alpha}(u, v)$, for all $\alpha \in \mathbb{R}$
- **P2:** W is symmetric with respect to the interchange: $(u, v) \mapsto (v, u)$
- **P3:** W is analytic in its variables near $u = v = 0$, such that $W = O(4)$

General structure of the nonlinearity W.

- If $W \in \mathbb{C}$ and property **P1** is satisfied, such that
$$W(u, \bar{u}, v, \bar{v}) = W\left(ue^{i\alpha}, \bar{u}e^{-i\alpha}, ve^{i\alpha}, \bar{v}e^{-i\alpha}\right), \quad \forall \alpha \in \mathbb{R}, \quad \text{then}$$
$$W = W(|u|^2, |v|^2, u\bar{v}).$$

- If $W \in \mathbb{R}$ and property **P1** is met, then
$$W = W(|u|^2, |v|^2, u\bar{v} + v\bar{u}).$$

- If $W \in \mathbb{R}$ and properies **P1-P3** are satisfied, then
$$W = W(|u|^2 + |v|^2, |u|^2|v|^2, u\bar{v} + v\bar{u}).$$

Special cases of the nonlinearity W.

$$W = \frac{a_1}{2}(|u|^4 + |v|^4) + a_2|u|^2|v|^2 + a_3(|u|^2 + |v|^2)(v\bar{u} + \bar{v}u) + \frac{a_4}{2}(v\bar{u} + \bar{v}u)^2$$

- $a_1, a_2 \neq 0$ and $a_3 = a_4 = 0$ represents a standard coupled-mode system for a sub-harmonic resonance, e.g. in the context of optical gratings.

  (C.M. de Sterke and J.E. Sipe, "Gap solitons", Progress in Optics, **33**, 203 (1994))

- $a_1 = a_3 = a_4 = 0$ system is integrable, inverse scattering is applied and is referred to as the massive Thirring model.

  (D.J. Kaup and A.C. Newell, "On the Coleman correspondence and the solution of the Massive Thirring model", Lett. Nuovo Cimento **20**, 325–331 (1977))

Spectral stability of the stationary solution.

- A stationary solution $u^*(x)$ of a dynamical system $\dot{u} = F(u)$ is **spectrally stable** if the spectrum of the linear operator obtained by linearizing $F(u)$ around u^* has no strictly positive real part.

- A stationary solution $u^*(x)$ of a dynamical system $\dot{u} = F(u)$ is **Lyapunov stable** if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $||u^*(x) - u(x,t)||_{t=0} < \delta$, then $||u^*(x) - u(x,t)|| < \epsilon$ for all t .

Spectral stability is a **necessary condition** for nonlinear stability.

Hamiltonian structure of the coupled mode system.

- We rewrite the coupled-mode system as a Hamiltonian system in complex-valued matrix-vector notations:

$$\frac{d\mathbf{u}}{dt} = J\nabla H(\mathbf{u}),$$

where $\mathbf{u} = (u, \bar{u}, v, \bar{v})^T$,

$$J = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} = -J^T,$$

and $H(u, \bar{u}, v, \bar{v}) = \int_{\mathbb{R}} h(u, \bar{u}, v, \bar{v}) dx$ is the Hamiltonian functional with the density:

$$h = W(u, \bar{u}, v, \bar{v}) - (v\bar{u} + u\bar{v}) + \frac{i}{2}(u\bar{u}_x - u_x\bar{u}) - \frac{i}{2}(v\bar{v}_x - v_x\bar{v}).$$

Conserved quantities of the coupled-mode system.

- The Hamiltonian $H(u, \bar{u}, v, \bar{v})$ is constant in time $t \geq 0$. Due to the gauge invariance, the coupled-mode system has another constant of motion $Q(u, \bar{u}, v, \bar{v})$, where

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2 \right) dx.$$

- Due to the translational invariance, the coupled-mode system has yet another constant of motion $P(u, \bar{u}, v, \bar{v})$, where

$$P = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v} \right) dx.$$

- In applications, the quantities Q and P are referred to as the power and momentum of the coupled-mode system.

Classes of Hamiltonian evolution equations.

- Linearized Hamiltonian PDE

$$\frac{dv}{dt} = JHv$$

- Nonlinear Schrödinger equation (NLS)

$$i\psi_t = -\psi_{xx} + U(x)\psi + F(|\psi|^2)\psi$$

Bounds on complex eigenvalues are given by the Pontryagin theorem.

- Korteweg-De Vries equation (KdV)

$$u_t + \partial_x(f(u) + u_{xx}) = 0$$

Bounds on complex eigenvalues are given by the Pontryagin theorem.

- Coupled Mode System(CM)

$$i(u_t + u_x) + v + \partial_{\bar{u}} W(u, \bar{u}, v, \bar{v}) = 0,$$

$$i(v_t - v_x) + u + \partial_{\bar{v}} W(u, \bar{u}, v, \bar{v}) = 0.$$

Can we apply Krein space theory for this case ?

General characterization of 1-D gap solitons

Stationary solutions of the coupled-mode system:

$$\begin{cases} u_{\text{st}}(x,t) = u_0(x+s)e^{i\omega t + i\theta} \\ v_{\text{st}}(x,t) = v_0(x+s)e^{i\omega t + i\theta} \end{cases}$$

- $(s, \theta) \in \mathbb{R}^2$ are arbitrary parameters and $-1 < \omega < 1$
- If $|u_0|, |v_0| \to 0$ as $|x| \to \infty$, then $u_0 = \bar{v}_0$
- Analytical expressions are available for homogeneous functions W. For example if $(a1 = 1, a2 = a3 = a4 = 0)$

$$u_0 = \frac{\sqrt{2(1-\omega)}}{(\cosh \beta x + i\sqrt{\mu} \sinh \beta x)}, \qquad \mu = \frac{1-\omega}{1+\omega}, \quad \beta = \sqrt{1-\omega^2}$$

- Explicit gap solitons are *stationary* solutions. *Traveling* gap solitons are only available implicitly except few special examples.

Linearized stability problem for 1-D gap solitons

- Standard linearization, e.g.

$$u(x,t) = e^{i\omega t}\left(u_0(x) + U_1(x)e^{\lambda t}\right)$$

- Eigenvalue problem

$$i\sigma H_\omega \mathbf{U} = \lambda \mathbf{U}, \quad \mathbf{U} \in \mathbb{C}^4,$$

where

$$H_\omega = D(\partial_x) + D^2 W[u_0(x)], \qquad \sigma = diag(1,-1,1,-1)$$

and $D(\partial_x)$ is the four-component Dirac operator in 1-D

$$D = \begin{pmatrix} \omega - i\partial_x & 0 & -1 & 0 \\ 0 & \omega + i\partial_x & 0 & -1 \\ -1 & 0 & \omega + i\partial_x & 0 \\ 0 & -1 & 0 & \omega - i\partial_x \end{pmatrix}$$

Structure of kernels of H_ω and L

- Due to the gauge and translational symmetries, the energy operator H_ω has at least a two-dimensional kernel with two eigenvectors:

$$\mathbf{U}_1 = \sigma \mathbf{u}_0(x), \qquad \mathbf{U}_2 = \mathbf{u}_0'(x).$$

- Due to the Hamiltonian structure, the linearized operator $i\sigma H_\omega$ has at least four-dimensional generalized kernel with two eigenvectors and two generalized eigenvectors.

$$\mathbf{U}_3 = \sigma \frac{\partial}{\partial \omega} \mathbf{u}_0(x), \qquad \mathbf{U}_4 = -\frac{1}{2}\, x\, \mathbf{D}^{-\mathbf{1}}\, \mathbf{u}_0(x).$$

(D.E. Pelinovsky, "Inertia law for spectral stability of solitary waves in coupled nonlinear Schrodinger equations", Proc. Roy. Soc. Lond. A **461**, 783–812 (2005))

The location of the continuous spectrum.

- The continuous spectrum for the linearized coupled-mode system can be found from the no-potential case $V(x) \equiv 0$.

- It consists of two pairs of symmetric branches on the imaginary axis $\lambda \in i\mathbb{R}$ for $|\mathrm{Im}(\lambda)| > 1 - \omega$ and $|\mathrm{Im}(\lambda)| > 1 + \omega$.

- In the potential case $V(x) \neq 0$, the continuous spectrum does not move, but the discrete spectrum appears.

- The discrete spectrum is represented by symmetric pairs or quartets of isolated non-zero eigenvalues and zero eigenvalue of algebraic multiplicity four for the generalized kernel of $i\sigma H_\omega$

Block-diagonalization of the stability problem

- Exists the orthogonal similarity transformation S that simultaneously block-diagonalizes the energy operator H_ω and the linearized Hamiltonian $L = i\sigma H_\omega$

$$S^{-1} H_\omega S = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \qquad S^{-1} i\sigma H_\omega S = i\sigma \begin{pmatrix} 0 & H_- \\ H_+ & 0 \end{pmatrix}.$$

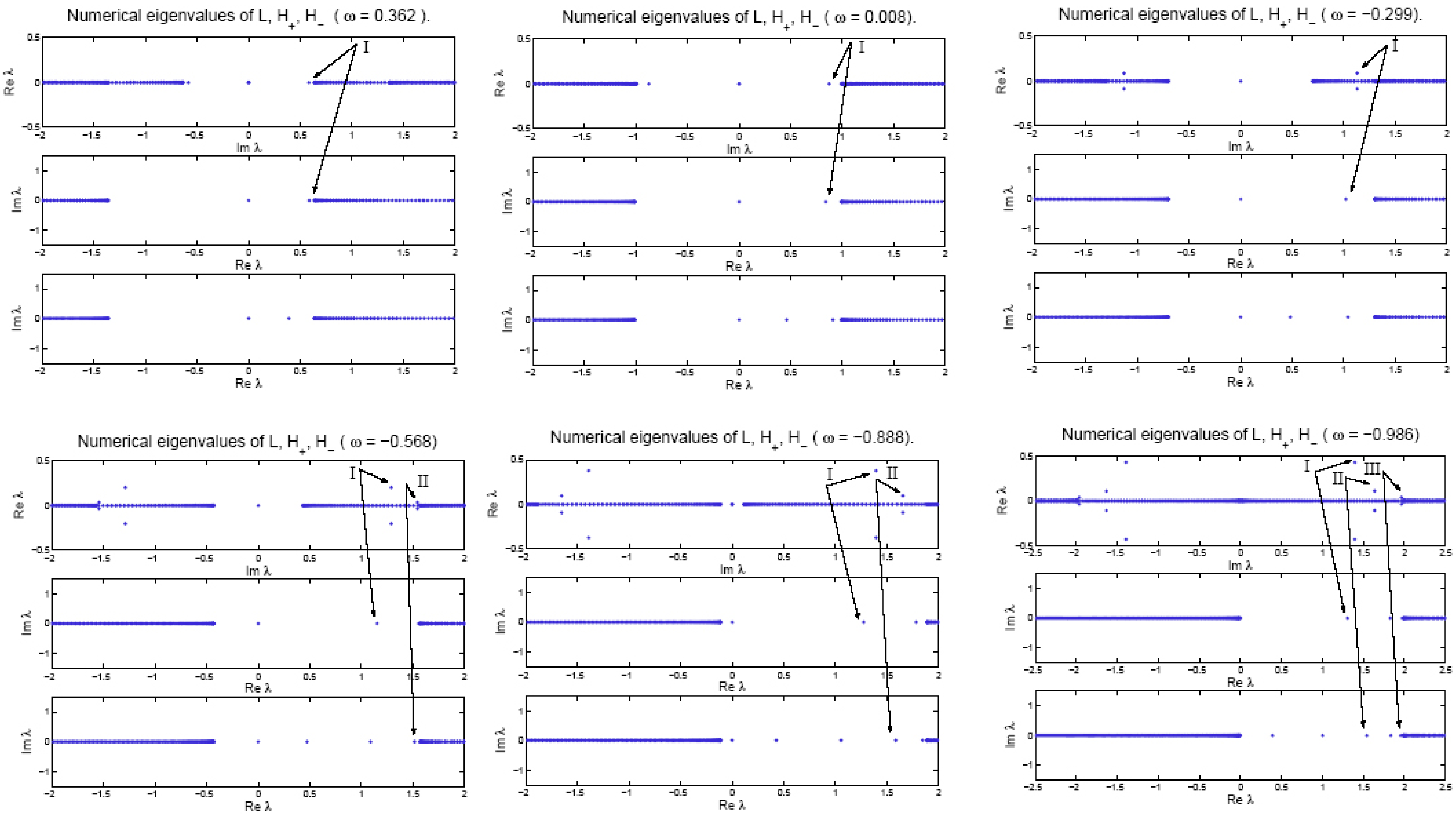
Where $H_\pm$ are two-by-two Dirac operators:

$$H_\pm = \begin{pmatrix} \omega - i\partial_x & \mp 1 \\ \mp 1 & \omega + i\partial_x \end{pmatrix} + V_\pm(x)$$

and

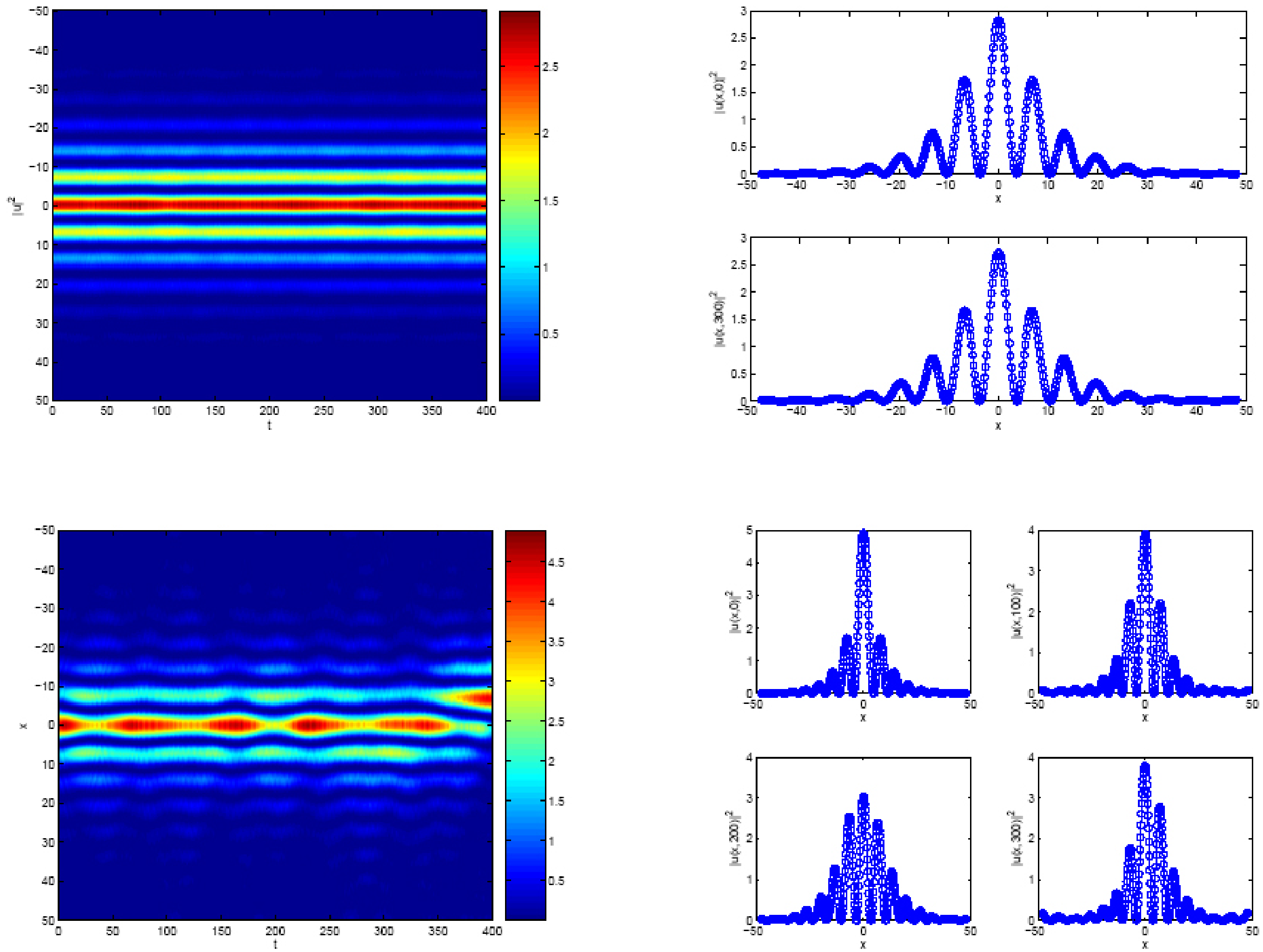
$$V_\pm = \begin{pmatrix} \partial^2_{\bar{u}_0 u_0} \pm \partial^2_{\bar{u}_0 \bar{v}_0} & \partial^2_{\bar{u}_0^2} \pm \partial^2_{\bar{u}_0 v_0} \\ \partial^2_{u_0^2} \pm \partial^2_{u_0 \bar{v}_0} & \partial^2_{\bar{u}_0 u_0} \pm \partial^2_{u_0 v_0} \end{pmatrix} W(u_0, \bar{u}_0, v_0, \bar{v}_0).$$

Eigenvalues and instability bifurcations



- Eigenvalues and instability bifurcations for the symmetric quadric potential W with $a_1 = 1$ and $a_2 = a_3 = a_4 = 0$.

Stable and unstable regime.



- Light propagation in the optical cable.

The Dirac equation in the vacuum.

- The conclusion that physical vacuum had nontrivial structure was first made by Dirac on the basis of the equation:

$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi(x) = 0$$

where $\gamma^{\mu} = (\gamma^0, \gamma)$ are γ-Dirac matrices $\partial_{\mu} = (\partial_0, \partial_1) = (\partial_t, \partial_x)$, m is a fermion mass and $\Psi(x)$ is a $4-$component spinor field.

- The nonlinear Dirac equation in vacuum

$$(i\gamma^{\mu}\partial_{\mu} - g(\overline{\Psi}\Psi))\Psi(x) = 0$$

where $g(0) = m > 0$

- We can rewrite this as a system

$$\begin{cases} i\partial_t\Psi_1 = \partial_x\Psi_2 + g(|\Psi_1|^2 - |\Psi_2|^2)\Psi_1, \\ i\partial_t\Psi_2 = -\partial_x\Psi_1 - g(|\Psi_1|^2 - |\Psi_2|^2)\Psi_2. \end{cases}$$

The most simple case of the nonlinearity $g(x) = x$.

- Potential function W takes the form

$$W = \frac{1}{2}(\Psi_1\overline{\Psi_2} + \overline{\Psi_1}\Psi_2)^2.$$

- System takes the form

$$\begin{cases} i(\partial_t + \partial_x)\Psi_1 + \Psi_2 = \partial_{\overline{\Psi_1}} W \\ i(\partial_t - \partial_x)\Psi_2 + \Psi_1 = \partial_{\overline{\Psi_2}} W. \end{cases}$$

- Stationary solutions can be found explicitly

$$\Psi_1^0(x) = \sqrt{Q(x)}\exp[-i\Theta(x)], \qquad \Psi_2^0(x) = \overline{\Psi_1^0(x)},$$

where

$$Q(x) = (1+\omega)\frac{(\cosh^2(\beta x) + \mu\sinh^2(\beta x))}{(\cosh^2(\beta x) - \mu\sinh^2(\beta x))^2},$$

$$\cos(\Theta) = \frac{\cosh(\beta x)}{\sqrt{(\cosh^2(\beta x) + \mu\sinh^2(\beta x))}}, \quad \beta = \sqrt{1-\omega^2}, \quad \mu = \frac{1+\omega}{1-\omega}.$$

Diagonalization and spectral analysis for the linearized

- Block-diagonalized energy operator $H_\omega = diag(H_+, H_-)$ where

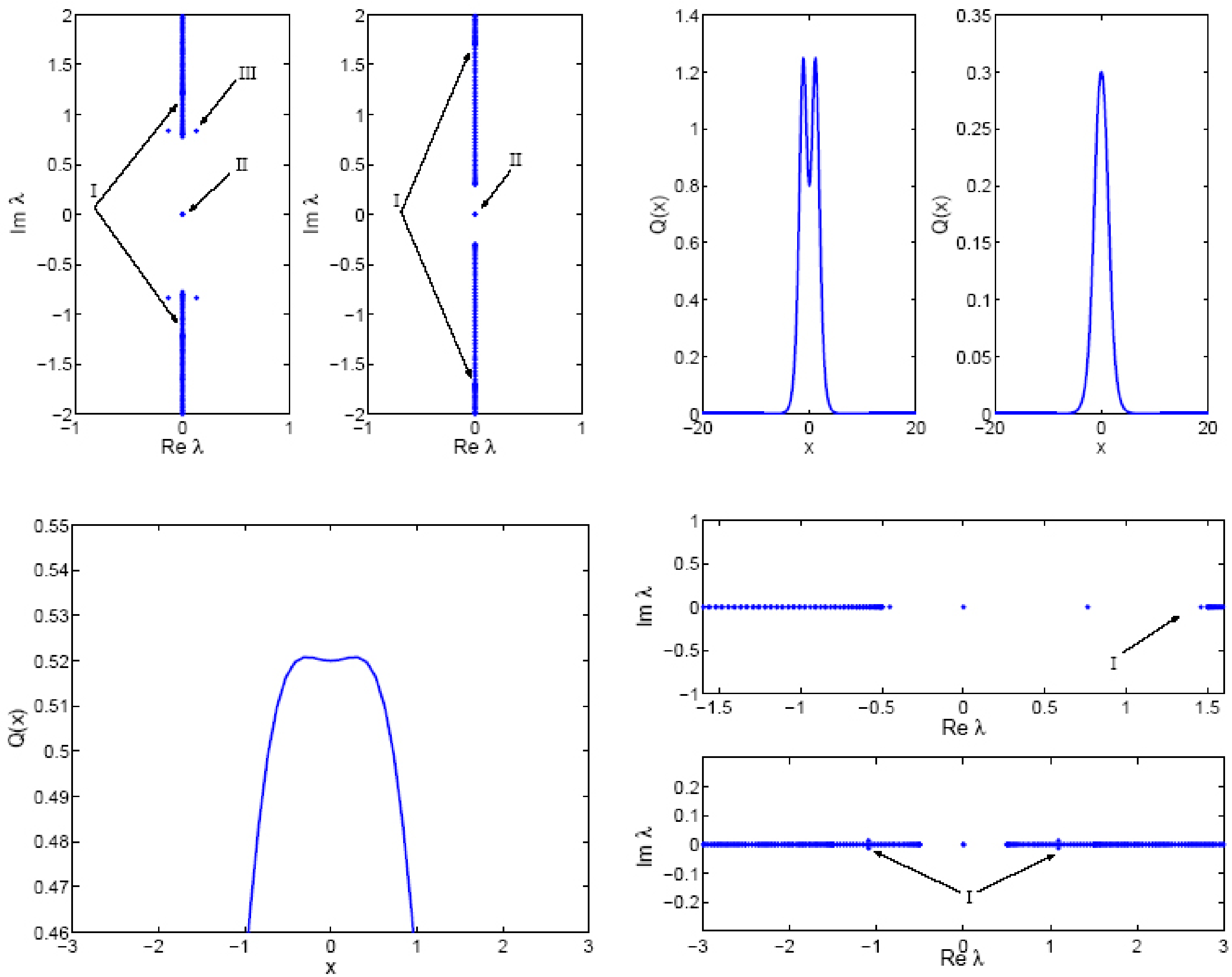
$$H_+ = \begin{pmatrix} -\omega - i\partial_x + 2|\Psi_0|^2 & \Psi_0^2 + 3\overline{\Psi_0}^2 - 1 \\ \overline{\Psi_0}^2 + 3\Psi_0^2 - 1 & -\omega + i\partial_x + 2|\Psi_0|^2 \end{pmatrix},$$

$$H_- = \begin{pmatrix} -\omega - i\partial_x & 1 - \Psi_0^2 - \overline{\Psi_0}^2 \\ 1 - \Psi_0^2 - \overline{\Psi_0}^2 & -\omega + i\partial_x \end{pmatrix}.$$

- Spectral analysis of the non self-adjoint operator L

$$L = i\,\sigma \begin{pmatrix} 0 & H_+ \\ H_- & 0 \end{pmatrix}.$$

Double-hump soliton and instability bifurcation



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