Positive operators in Krein spaces similar to self-adjoint operators in Hilbert spaces

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Set $[\,\cdot\,,\cdot\,] = \langle J\,\cdot\,,\,\cdot\,\rangle$. Then $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]\big)$ is a Krein space.

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With $\mathcal{H}_\pm = (I \pm J_1)\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_-[\dot{+}]\mathcal{H}_+ \quad \text{is a fundamental decomposition}$$

$P_\pm = \frac{1}{2}(I \pm J_1)$ are the corresponding fundamental projections.

Examples.

Let $n \geq 1$, $w \in L_{\mathrm{loc}}(\mathbb{R}^n)$, $w > 0$ a.e. on $\mathbb{R}^n$.

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or

$$(Jf)(x) = f(\mathsf{M}x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

M is $n \times n$ matrix such that $\mathsf{M}^2 = \mathsf{I}$ and $w(\mathsf{M}x) = w(x), x \in \mathbb{R}^n$.

For example, $(Jf)(x) = f(-x)$, $x \in \mathbb{R}^n$ and $w = 1$.

Albaverio&Kuzhel call this J the space parity operator.

Definition. Let $(\mathcal{H}, [\,\cdot\,,\cdot\,])$ be a Krein space. In this talk an operator $A : \operatorname{dom}(A) \to \mathcal{H}$ is *positive* in $(\mathcal{H}, [\,\cdot\,,\cdot\,])$, (or [positive] for short) if the following three conditions are satisfied

1. $\varrho(A) \neq \emptyset$.

2. A is [self-adjoint] in $(\mathcal{H}, [\,\cdot\,,\cdot\,])$.
   $\Leftrightarrow$ JA ⟨self-adjoint⟩ in the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle)$.

3. $[Ax, x] > 0$ for all $x \in \operatorname{dom}(A) \setminus \{0\}$.
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(d) The following two statements hold.

(∞) $\exists\ \mu > 0$ and a [positive] homomorphism W on $\mathcal{H}$ such that $W \operatorname{dom}\big((JA)^\mu\big) \subseteq \operatorname{dom}\big((JA)^\mu\big)$.

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The operators from Theorem 1 we denote by [p]s⟨sa⟩.

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With $p(x) = |x|^\beta$, $\beta < 1$, set

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Here $\mathsf{M} = \mathsf{M}^{-1}$ is an $n \times n$ generalized permutation matrix whose nonzero entries are either 1 or -1. For example

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Then $A = -J\,\Delta$ is [p]s⟨sa⟩. (C&Najman)

Form bounded perturbations.

Theorem 2.(C&Najman, Jonas) $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]\big)$ is a Krein space; A is [p]s⟨sa⟩ operator; a is the closure of the form $[A\,\cdot\,,\,\cdot\,]$.

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Let v be a closed symmetric form in $\mathcal{H}$ such that:

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- $$c_- \, a(x) \le v(x) \le c_+ \, a(x), \qquad x \in \mathrm{dom}(a),$$
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Faddeev&Shterenberg, Karabash&Malamud, Karabash&Kostenko

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Hence, for all such potentials,

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An example (Kato): $n = 3$, $a_j \in \mathbb{R}^3$, $c_j > 0$,

$$q(x) = -\sum_{j=1}^{k} \frac{c_j}{|x - a_j|^2} \quad \text{and} \quad \kappa < \frac{1}{4\sum c_j}.$$

The class of operators $\big(\textbf{[p]s}\langle\textbf{sa}\rangle\big)_{\infty,\rho}$

Definition.

Let A be a [self-adjoint] operator in a Krein space $\big(\mathcal{H},[\,\cdot\,,\cdot\,]\big)$.

$$A \text{ is } \big(\text{[p]s}\langle\text{sa}\rangle\big)_{\infty,\rho}$$

that is A is [positive] and similar to ⟨self-adjoint⟩ in $\mathbb{E}_\rho = \{z \in \mathbb{C} : |z| > \rho\}$ if

- $\mathcal{H} = \mathcal{H}_0[\dot{+}]\mathcal{H}_\infty$.
- $\mathcal{H}_0$ and $\mathcal{H}_\infty$ are invariant under A.
- $A_0 = A|_{\mathcal{H}_0}$ is bounded and $\sigma(A_0) \subset \{z \in \mathbb{C} : |z| \le \rho\}$.
- $A_\infty = A|_{\mathcal{H}_\infty}$ is [p]s⟨sa⟩.

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- If $\imath \subset (\rho, +\infty)$, then $\big(E(\imath)\mathcal{H}, [\,\cdot\,,\cdot\,]\big)$ is a Hilbert space.

  If $\imath \subset (-\infty, -\rho)$, then $\big(E(\imath)\mathcal{H}, -[\,\cdot\,,\cdot\,]\big)$ is a Hilbert space.

Theorem 3.(C&Jonas) Assume

- A is [p]s⟨sa⟩.

- B is [self-adjoint] and bounded.

Then $A + B$ is $\big($[p]s⟨sa⟩$\big)_{\infty,\rho}$ for some $\rho > 0$.

We also give an estimate for ρ.

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Under these conditions ⟨self-adjoint⟩ operators in $\mathcal{H}$ can be associated with the expression

$$\frac{1}{|w|}\left(-\frac{d}{dx}\Big(p\,\frac{d}{dx}\Big) + q\right)$$

Let S be such an operator in the Hilbert space $\mathcal{H} = L^2(\imath; |w|)$.

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Then $A = JS$ is $\big(\text{[p]s}\langle\text{sa}\rangle\big)_{\infty,\rho}$.

Here ρ might be hard to calculate.

For the special case $\imath = \mathbb{R}$, $p = 1$, $q \in L^\infty(\mathbb{R})$
and $w(x) = \operatorname{sgn} x$ we calculated that

$$\rho < 7.903 \ \|q\|_\infty.$$

For example:

$$A = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} - \sin x\right)$$

$$\sigma(A) \subset \mathbb{R} \cup \left\{z \in \mathbb{C} : |z| < 7.903\right\}$$

and

$(-\infty, -7.903]$ are spectral points of negative type and

$[7.903, +\infty)$ are spectral points of positive type.

The end

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The following statements are equivalent

- $$\left|\int_{\mathbb{R}} q(x)\,|u(x)|^2 dx\right| \le C\left(\int_{\mathbb{R}} |u'(x)|^2 dx + \int_{\mathbb{R}} |u(x)|^2 dx\right)$$

- $$\sup_{x\in\mathbb{R}} \int_x^{x+1} \Big(|\Gamma(\xi)|^2 + |\gamma(\xi)|\Big) d\xi < +\infty$$

where

$$\Gamma(\xi) = \int_{\mathbb{R}} \operatorname{sgn}(\xi - t) e^{-|\xi-t|} q(t)dt, \quad \gamma(\xi) = \int_{\mathbb{R}} e^{-|\xi-t|} q(t)dt.$$